Higher resonance varieties

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Abstract We present some new results about the resonance varieties of matroids and hyperplane arrangements. Though these have been the objects of ongoing study, most work so far has focussed on cohomological degree 1. We show that certain phenomena become apparent only by considering all degrees at once.

Key words: hyperplane arrangement, matroid, resonance variety

1 Introduction

Resonance varieties are a cohomological invariant that first appeared in the study of the cohomology of one-dimensional fundamental group representations. Though they were first considered for topological spaces, they are algebraic in nature, and they may be defined for any (differential) graded-commutative algebra [31, 32].

The resonance varieties associated with the complement of a complex hyperplane arrangement are an interesting special case, and we mention in particular the surveys [18, 36] for their description of the history and motivation. Here, the underlying topological space is a complex, quasiprojective

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variety; however, it follows from the Brieskorn-Orlik-Solomon Theorem [23] that the resonance varieties depend only on combinatorial data coming from a matroid.

With that in mind, it is tempting to ask for a formula or an efficient algorithm that expresses the resonance varieties in terms of the matroid. However, this seems to be a difficult problem. In cohomological degree 1, Falk and Yuzvinsky [19] have given a characterization, building on work of Libgober and Yuzvinsky [22] as well as Falk [17]. In degree greater than 1, there has been some progress (in particular [4, 9]), but comparatively little is known. By way of contrast, Papadima and Suciu gave a closed formula for the resonance varieties of exterior Stanley-Reisner rings (and right-angled Artin groups) in [26], building on work of [1]. Although the situation for matroids has some similarities, an analogous definitive result is so far out of reach.

Our main goal here is to develop some basic tools systematically. We consider the behaviour of resonance varieties of matroids and arrangements under such constructions as weak maps, Gale duality, and the deletion/contraction constructions. Combining these basic ingredients allows us to compute the resonance varieties of some graphic arrangements explicitly. A broader range of phenomena appear in moving from degree 1 to 2: as an example, we find a straightforward way to make hyperplane arrangements for which the Milnor fibre F has nontrivial monodromy eigenspaces in $H^2(F,\mathbb{C})$.

A hyperplane arrangement is a matroid realization over $\mathbb{C}$. Some results about resonance from the literature are known for all matroids, while others depend on realizability. Our approach is combinatorial, so matroids (rather than arrangements) seem to be the appropriate objects for this paper. It was recently shown in [28] that realizability imposes a non-trivial qualitative restriction on resonance varieties, at least in positive characteristic. This encourages us to keep track of the role of realizability.

1.1 Outline

The paper is organized as follows. We begin by recalling the definition of the Orlik-Solomon algebra, viewed as a matroid invariant. We would like to make use of the naturality of the construction; however, not all weak maps of matroids induce homomorphisms of Orlik-Solomon algebras. In §2.3 we impose a condition on weak maps to define a category $\mathcal{M}$ of matroids on which the Orlik-Solomon construction is functorial.

In §3, we define resonance varieties and review known results about some qualitative properties that distinguish the resonance varieties of Orlik-Solomon algebras from the general case. For example, for a matroid M of rank ℓ, the resonance varieties are known to satisfy

$$\mathcal{R}^p(\mathsf{M}) \subseteq \mathcal{R}^{p+1}(\mathsf{M})$$

for $0 \leq p < \ell$ [15]. At least in the realizable, characteristic-zero case, resonance varieties are unions of linear subspaces. In §3.4, we construct subspace arrangements $\mathcal{S}^p(\mathsf{M})$ that contain them, based on a result of Cohen, Dimca, and Orlik [7]. We find that in some interesting cases this upper envelope is tight: i.e., $\mathcal{R}^p(\mathsf{M}) = \mathcal{S}^p(\mathsf{M})$.

In §4, we examine the effect of standard matroid operations on resonance. Some results are known, some folklore, and others new. We use these to compute some examples and find that certain special properties of components of $\mathcal{R}^p(\mathsf{M})$ for $p = 1$ no longer hold for $p \geq 2$.

In the last section, we revisit the combinatorics of multinets and singular subspaces in terms of maps of Orlik-Solomon algebras as another attempt to characterize components of resonance varieties. The results are inconclusive, although the last example strongly suggests that some interesting combinatorics remains to be uncovered.

2 Background

2.1 Arrangements and matroids

We refer to the books of Orlik and Terao [24] and Oxley [25] for basic facts about hyperplane arrangements and matroids, respectively. If M is a matroid on the set $[n] := \{1, 2, \ldots, n\}$ and $\Bbbk$ is a field, let $V = \Bbbk^n$, a vector space with a distinguished basis we will call $e_1, \ldots, e_n$. The Orlik-Solomon algebra $A_\Bbbk(\mathsf{M})$ is the quotient of an exterior algebra $E := \Lambda(V)$ by an ideal $I = I(\mathsf{M})$ generated by homogeneous relations indexed by circuits in M. More explicitly, let ∂ be the derivation on E defined by $\partial(e_i) = 1$ for all $1 \leq i \leq n$. Then I is generated by

$$\{\partial(e_C) \colon \text{circuits } C \subseteq [n] \text{ of } \mathsf{M}\}, \tag{1}$$

where $e_C := \prod_{i \in C} e_i$, with indices taken in increasing order. We will omit the M or $\Bbbk$ from the notation $A_\Bbbk(\mathsf{M})$ where no confusion arises. If $i \in \mathsf{M}$ is a loop, then $C = \{i\}$ is a circuit and $A(\mathsf{M}) = 0$.

We regard hyperplane arrangements as linear representations of loop-free matroids. For us, an arrangement $\mathcal{A}$ over a field F is an ordered n-tuple of (nonzero) linear forms $(f_1, \ldots, f_n)$, where $f_i \in W^*$ for $1 \leq i \leq n$, and W is a vector space over F. We let $\mathsf{M}(\mathcal{A})$ denote the matroid on $[n]$ whose dependent sets index the linear dependencies of the f_i's. In particular, our arrangements are all central, and we explicitly allow repeated hyperplanes.

If $\mathcal{A}$ is an arrangement, let $H_i = \ker(f_i)$, a hyperplane in W, for $1 \leq i \leq n$. Let $M(\mathcal{A}) = W - \bigcup_{i=1}^n H_i$, and $U(\mathcal{A}) = \mathbb{P}W - \bigcup_{i=1}^n \mathbb{P}H_i$. If $\mathcal{A}$ is an arrangement, we abbreviate $A_\Bbbk(\mathcal{A}) := A_\Bbbk(\mathsf{M}(\mathcal{A}))$. If $F = \mathbb{C}$, the complement $M(\mathcal{A})$ is a complex manifold, and the Brieskorn-Orlik-Solomon Theorem states that $A_\Bbbk(\mathcal{A}) \cong H^*(M(\mathcal{A}), \Bbbk)$ as graded algebras.

2.2 Projectivization

Suppose $A(\mathsf{M}) = E/I$ is the Orlik-Solomon algebra of a matroid M. Since $\partial^2 = 0$, it follows $\partial(I) = 0$, so ∂ induces a derivation on A as well, which we denote by ∂_A. Let

$$\overline{V} = \ker(\partial|_V) = \Big\{v \in \Bbbk^n \colon \sum_{i=1}^n v_i = 0\Big\},$$

and let $\overline{A}(\mathsf{M})$ denote the subalgebra of A generated by $\overline{V}$.

Lemma 2.1. *We have* $\overline{A}(\mathsf{M}) = \ker \partial_A$.

Proof. Clearly $\overline{A}(\mathsf{M}) \subseteq \ker \partial_A$. By [24, Lemma 3.13], the chain complex (A, ∂) is exact, so if $\partial_A(x) = 0$, then $x = \partial_A(y)$ for some $y \in A$. If $e_I = e_{i_1} \cdots e_{i_k} \in E$, then $\partial(e_I) = (e_{i_1} - e_{i_k})(e_{i_2} - e_{i_k}) \cdots (e_{i_{k-1}} - e_{i_k})$, which implies $\partial(e_I)$ is in the subalgebra generated by $\overline{V}$. By applying ∂_A to a representative in E for y, we see $x \in \overline{A}(\mathsf{M})$. ☐

Together with the exactness of (A, ∂_A), this gives a short exact sequence of graded $\Bbbk$-modules,

$$0 \longrightarrow \overline{A}(\mathsf{M}) \longrightarrow A(\mathsf{M}) \xrightarrow{\partial} \overline{A}(\mathsf{M})[-1] \longrightarrow 0. \tag{2}$$

Of course, if $\mathsf{M} = \mathsf{M}(\mathcal{A})$ where $\mathcal{A}$ is a complex arrangement, this sequence has a well-known origin: the quotient map $M(\mathcal{A}) \to U(\mathcal{A})$ makes $M(\mathcal{A})$ a split $\mathbb{C}^*$-bundle over $U(\mathcal{A})$, so the induced algebra homomorphism $H^*(U(\mathcal{A}), \Bbbk) \to H^*(M(\mathcal{A}), \Bbbk)$ is injective. In fact, under the isomorphism $H^*(M(\mathcal{A}), \Bbbk) \cong A(\mathcal{A})$, the Gysin map is identified with ∂_A, and $\overline{A}(\mathcal{A}) \cong H^*(U(\mathcal{A}), \Bbbk)$: see [6] or [13, §6.1] for details. With this in mind, we will call $\overline{A}(\mathsf{M})$ the *projective Orlik-Solomon algebra* even if M does not have a complex realization.

2.3 A category of matroids

We would like to make use of maps of Orlik-Solomon algebras, so it will be useful to have a functorial construction. For this, we recall the definition of a weak map of matroids from [34, Ch. 9]. If M_1 and M_2 are matroids on sets S_1 and S_2, respectively, we add a disjoint loop "0" to M_i to make a matroid M_i^+, for $i = 1, 2$.

Definition 2.2. A *weak map* $f \colon \mathsf{M}_1 \to \mathsf{M}_2$ is a map of sets $f \colon S_1 \cup \{0\} \to S_2 \cup \{0\}$ with the following properties: $f(0) = 0$, and for all $I \subset S_1$, if $f|_I$ is injective and $f(I)$ is independent in M_2^+, then I is independent in M_1.

We will say a weak map $f\colon \mathsf{M}_1 \to \mathsf{M}_2$ is *non-degenerate* if $f^{-1}(0) = \{0\}$ and *complete* if, for every circuit C of M_1, we have $\left|C \cap f^{-1}(0)\right| \neq 1$. Clearly, non-degenerate weak maps are complete.

Proposition 2.3. *If $f\colon \mathsf{M}_1 \to \mathsf{M}_2$ and $g\colon \mathsf{M}_2 \to \mathsf{M}_3$ are complete weak maps, so is $g \circ f$. If f and g are non-degenerate weak maps, so is $g \circ f$.*

Proof. Weak maps are closed under composition, so the second assertion is trivial, and it is enough to check that the composite of complete maps is complete. Suppose instead that there is a circuit C in M_1 and a unique element $i \in C$ for which $g \circ f(i) = 0$. Since f is complete, $f(i) \neq 0$. Then $f(i)$ is contained in a circuit $C' \subseteq f(C)$ of M_2. By assumption, $j = f(i)$ is the only element of C' which $g(j) = 0$. But g is complete, a contradiction. ☐

In view of the previous result, matroids on finite sets form a category with morphisms taken to be the complete weak maps, which we will denote by $\mathcal{M}$. Let $\overline{\mathcal{M}}$ denote the (wide) subcategory whose morphisms are non-degenerate weak maps. Our hypotheses (complete and non-degenerate) are designed to make the Orlik-Solomon algebra and its projective version functorial, respectively.

First, we note a categorical formality. If $f\colon \mathsf{M}_1 \to \mathsf{M}_2$ is a complete weak map and $f(S_1) \supseteq S_2$, then it is easy to see f is an epimorphism. While we avoid trying to characterize epimorphisms, here is a convenient necessary condition.

Lemma 2.4. *If $f\colon \mathsf{M}_1 \to \mathsf{M}_2$ is an epimorphism in $\mathcal{M}$, then any element $i \in S_2 - f(S_1)$ is either a loop or it is parallel to an element of $f(S_1)$.*

Proof. Suppose instead that some $i \in S_2 - f(S_1)$ is neither a loop nor parallel to an element of $f(S_1)$. Define two maps $g, h\colon \mathsf{M}_2 \to \mathsf{U}_{1,1}$: let $g(j) = 0$ for all $j \in S_2$ and $h(j) = 0$ for all $j \neq i$, but $h(i) = 1$. Then g and h are complete weak maps, but $g \circ f = h \circ f$, so f is not an epimorphism. ☐

By the obvious action on the distinguished basis, a weak map $f\colon \mathsf{M}_1 \to \mathsf{M}_2$ induces a linear map $V(\mathsf{M}_1) \to V(\mathsf{M}_2)$ which we will also denote by f.

Lemma 2.5. *Suppose $f\colon \mathsf{M}_1 \to \mathsf{M}_2$ is a complete weak map of matroids. Then $\Lambda(f)$ induces a well-defined map of $\Bbbk$-algebras, $A(\mathsf{M}_1) \to A(\mathsf{M}_2)$. If f is also non-degenerate, then it restricts to a map of projective Orlik-Solomon algebras.*

Proof. Write $A(\mathsf{M}_i) = E_i/I_i$ for $i = 1, 2$, and consider the homomorphism $\Lambda(f)\colon E_1 \to E_2$. For any circuit C in M_1, we consider two cases. If $0 \notin f(C)$, then $f(C) \cap S_2$ is dependent, so the image of $\Lambda(f)(\partial(e_C)) \in I_2$. If $0 \in f(C)$, by completeness, $f(e_i) = f(e_j)$ for some distinct $i, j \in C$, so $\Lambda(f)(\partial(e_C)) = 0$. So $\Lambda(f)(I_1) \subseteq I_2$, as required.

If, moreover, f is non-degenerate, we see $\partial|_{V(\mathsf{M}_2)} \circ f = f \circ \partial|_{V(\mathsf{M}_1)}$, by evaluating on the distinguished basis. Then $\operatorname{im}(f|_{\overline{V}(\mathsf{M}_1)}) \subseteq \overline{V}(\mathsf{M}_2)$. Since $\overline{A}(\mathsf{M}_1)$ is generated in degree 1, the image of $\Lambda(f)|_{\overline{A}(\mathsf{M}_1)}$ lies in $\overline{A}(\mathsf{M}_2)$. ☐

If f is a complete weak map, put $A(f) = \Lambda(f) \colon A(\mathsf{M}_1) \to A(\mathsf{M}_2)$. If f is also non-degenerate, let $\overline{A}(f) \colon \overline{A}(\mathsf{M}_1) \to \overline{A}(\mathsf{M}_2)$ denote the restriction.

Theorem 2.6. *A and $\overline{A}$ are functors from $\mathcal{M}$ and $\overline{\mathcal{M}}$, respectively, to the category of graded-commutative $\Bbbk$-algebras. Moreover, A preserves epimorphisms.*

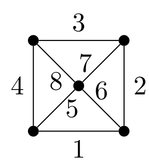
Proof. If f is a morphism, $A(f)$ is determined by its action in degree 1, where A obviously preserves composition, so A is functorial. Now suppose f is an epimorphism. The algebra $A(\mathsf{M}_2)$ is spanned by monomials e_I, where I is independent.

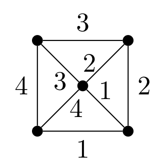
If $I \subseteq \mathrm{im}(f)$, there exists a subset $J \subseteq f^{-1}(I)$ with $|J| = |I|$. If not, by Lemma 2.4, we may replace some elements of I with parallel elements to form a set $I' \subseteq \mathrm{im}(f)$ and find J as above with $f(J) = I'$.

Since $e_i = e_j$ for parallel elements in $A(\mathsf{M}_2)$, in both cases we have $A(f)(e_J) = e_I$. So $A(f)$ is a surjective ring homomorphism. $\square$

Example 2.7. The map $f \colon \mathsf{U}_{2,3} \to \mathsf{U}_{2,2}$ given by $f(i) = i$ for $i = 1, 2$ and $f(3) = 0$ is a weak map. Taking $C = [3]$, we see f fails to be complete, and $\Lambda(f)$ fails to give a map of Orlik-Solomon algebras.

On the other hand, if M is the matroid of the graph

$G =$ 

, $\quad \mathsf{M}(G) \to \mathsf{U}_{3,4}$: 

(3)

then the map from the edges of G to the set $[4]$ given by the edge labels on the right defines a non-degenerate weak map $\mathsf{M}(G) \to \mathsf{U}_{3,4}$, because G contains no three-cycles with distinct edge labels. $\Diamond$

We will end this section with two easy but useful observations. Suppose M is a matroid on $[n]$ and π is a partition of $[n]$ into k parts. Let $p_\pi \colon [n] \to [k]$ be the map that sends i to s whenever $i \in \pi_s$.

Proposition 2.8. *$p_\pi \colon \mathsf{M} \to \mathsf{U}_{2,k}$ is a morphism of $\overline{\mathcal{M}}$ if and only if, whenever i and j are parallel in M, i and j are in the same block of π.*

Definition 2.9. If M is a matroid on $[n]$, we define an equivalence relation on $[n]$ by letting $i \sim j$ if and only if $\{i, j\}$ is dependent. The *simplification* of M, denoted M_s is, by definition, the induced matroid on the equivalence classes. The natural map $s \colon \mathsf{M} \to \mathsf{M}_s$ is a morphism of $\overline{\mathcal{M}}$.

The map $A(s) \colon A(\mathsf{M}) \to A(\mathsf{M}_s)$ is easily seen to be an isomorphism, where $I(\mathsf{M})$ contains relations $e_i - e_j$ if $\{i, j\}$ is dependent in M.

Remark 2.10. Clearly the complement of a hyperplane arrangement is unaffected by the presence of repeated hyperplanes, so for topological purposes there is no loss in assuming that the underlying matroid is simple. However, we will make some use of the fact that the isomorphism $A(s)$ is not an equality in Theorem 4.5. ◊

3 Resonance varieties

Now suppose $E = \Lambda(V)$ is the exterior algebra over a (finite-dimensional) $\Bbbk$-vector space V. Suppose A and B are graded E-modules.

3.1 Definitions

For any $v \in V$, we have $v \cdot v = 0$, so there is a cochain complex of $\Bbbk$-modules, $(A, \cdot v)$, in which the differential is by right-multiplication by v. This construction is natural, in the sense that if $f\colon A \to B$ is a graded E-module homomorphism, then for any $v \in V$,

$$f\colon (A, \cdot v) \to (B, \cdot f(v)) \tag{4}$$

is clearly a homomorphism of cochain complexes.

The resonance varieties of A are defined for all integers $p, d \geq 0$ to be

$$\mathcal{R}^p_d(A) = \{v \in V\colon \dim_{\Bbbk} H^p(A, \cdot v) \geq d\},$$

and we abbreviate $\mathcal{R}^p(A) := \mathcal{R}^p_1(A)$. We note that our definition varies slightly from the usual one (see, e.g., [27]) in that we do not assume either that A itself is a $\Bbbk$-algebra or that $V = A^1$. The modules of greatest interest are, in fact, algebras $A = A(\mathsf{M}) = E/I$; however, we do allow I to contain relations of degree 1, accommodating parallel elements in M. We suggest distinguishing the two parameters by referring to $\mathcal{R}^p_d(A)$ with $p > 1$ as "higher" resonance, versus "deeper" for $d > 1$. Our focus here is on the former.

For any nonzero $v \in V$, clearly $v \in \mathcal{R}^p_d(A)$ if and only if $\lambda v \in \mathcal{R}^p_d(A)$ for any $\lambda \in \Bbbk^\times$, so they determine projective subvarieties of $\mathbb{P}V$.

3.2 Resonance of Orlik-Solomon algebras

From now on, we restrict our attention to Orlik-Solomon algebras, and abbreviate: $\mathcal{R}^p(\mathsf{M}) := \mathcal{R}^p(A(\mathsf{M}))$ and $\mathcal{R}^p(\mathcal{A}) := \mathcal{R}^p(\mathsf{M}(\mathcal{A}))$ for matroids M and arrangements $\mathcal{A}$, respectively. If G is a graph, let $\mathsf{M}(G)$ denote its matroid, and $\mathcal{R}^p(G) := \mathcal{R}^p(\mathsf{M}(G))$.

First we mention some properties of resonance varieties that specific to Orlik-Solomon algebras. One such feature is a nestedness property discovered by Eisenbud, Popescu and Yuzvinsky [15] and studied further in [4, 11].

Theorem 3.1. *Let* M *be a matroid of rank* ℓ. *Then*

$$\{0\} \subseteq \mathcal{R}^0(\mathsf{M}) \subseteq \mathcal{R}^1(\mathsf{M}) \subseteq \cdots \subseteq \mathcal{R}^\ell(\mathsf{M}) \subseteq \overline{V}. \tag{5}$$

Proof. The inclusions $\mathcal{R}^p(\mathsf{M}) \subseteq \mathcal{R}^{p+1}(\mathsf{M})$ for $0 \leq p \leq \ell - 1$ were proven in [15, Thm. 4.1(b)]. The authors work with arrangements, but their arguments apply to all matroids. The inclusion $\mathcal{R}^p(\mathsf{M}) \subseteq \overline{V}$ for all $p \geq 0$ is due to Yuzvinsky, [35, Prop. 2.1]. □

Another is the following. By contrast, this result depends on complex geometry and a result due to Arapura [2]: for a full explanation, we refer to [14].

Theorem 3.2. *Let* $\mathcal{A}$ *be a complex hyperplane arrangement, and* $\Bbbk$ *a field of characteristic zero. Then* $\mathcal{R}^p_{\Bbbk}(\mathcal{A})$ *is a union of linear components, for* $0 \leq p \leq \operatorname{rank}(\mathcal{A})$.

Remark 3.3. In characteristic zero, then, resonance varieties of realizable matroids are subspace arrangements. Falk [20] has shown that this is not in general the case for $\operatorname{char} \Bbbk \neq 0$: see also [18, Ex. 4.24]. So even for hyperplane arrangements, the resonance varieties depend on the characteristic of the field (unlike the Orlik-Solomon algebra itself). For a striking application of resonance in characteristic 3, we refer to Papadima and Suciu [28]. ◊

A component W of a resonance variety is called *essential* if $W \cap (\Bbbk^\times)^n \neq \emptyset$.

Question 3.4. Assume $\operatorname{char} \Bbbk = 0$. Then the components of $\mathcal{R}^1(\mathsf{M})$ are linear for any matroid M [22, Cor. 3.6]. Is $\mathcal{R}^p_{\Bbbk}(\mathsf{M})$ a union of linear components for all matroids M, for $p > 1$?

The next result, due to Libgober and Yuzvinsky [22], is a qualitative property of $\mathcal{R}^1(\mathsf{M})$ which is both special to matroids and, we will see, to degree $p = 1$.

Theorem 3.5. *Assume* $\operatorname{char} \Bbbk = 0$. *If* $\mathcal{R}^1(\mathsf{M})$ *contains a component* W *of dimension* $k > 0$, *there is an injective homomorphism* $\overline{A}(\mathsf{U}_{2,k+1}) \to \overline{A}(\mathsf{M})$. *Conversely, the image of such a homomorphism in degree* 1 *lies in* $\mathcal{R}^1(\mathsf{M})$.

Proof. Multiplication in $\overline{A}^1(\mathsf{U}_{2,k+1})$ is zero, so this is just a reformulation of the following result from [22]: if W is a component of $\mathcal{R}^1(\mathsf{M})$, then for any $v, w \in W$, we have $vw = 0$. □

Next, we see that questions of resonance can be reduced to the projective Orlik-Solomon algebra, via the short exact sequence (2).

Lemma 3.6. *For any matroid* M *on* $[n]$ *and* $v \in \overline{V}$*, there is a short exact sequence of cochain complexes*

$$0 \longrightarrow (\overline{A}(\mathsf{M}), v) \longrightarrow (A(\mathsf{M}), v) \xrightarrow{\partial} (\overline{A}(\mathsf{M}), v)[-1] \longrightarrow 0. \qquad (6)$$

If $\operatorname{char} \Bbbk \nmid n$*, the sequence* (6) *is split.*

Proof. The inclusion $\overline{A}(\mathsf{M}) \to A(\mathsf{M})$ makes $A(\mathsf{M})$ an $\overline{A}(\mathsf{M})$-module. Using Lemma 2.1, it is easily checked that ∂ is a $\overline{A}(\mathsf{M})$-module homomorphism. With this, we see that the maps in the sequence (2) commute with multiplication by v.

If n is nonzero in $\Bbbk$, (left) multiplication by $\frac{1}{n}\sum_{i=1}^{n} e_i$ gives a right inverse to ∂, proving the second assertion. □

With this, we see that the resonance of A and $\overline{A}$ differ only by a trivial factor.

Proposition 3.7. *Let* M *be a matroid on* $[n]$ *of rank* ℓ*. If* $\operatorname{char} \Bbbk \nmid n$*, then* $\mathcal{R}^p(\overline{A}(\mathsf{M})) = \mathcal{R}^p(A(\mathsf{M}))$ *for all* $0 \le p \le \ell - 1$*, and* $\mathcal{R}^\ell(A(\mathsf{M})) = \mathcal{R}^{\ell-1}(A(\mathsf{M}))$*. For all* $d \ge 0$*, we also have*

$$\mathcal{R}^p_d(A(\mathsf{M})) = \bigcup_{j \le d} \mathcal{R}^p_j(\overline{A}(\mathsf{M}) \cap \mathcal{R}^{p-1}_{d-j}(\overline{A}(\mathsf{M})) \qquad (7)$$

Proof. The equality (7) is a direct consequence of Lemma 3.6. In particular, for $d = 1$, we have

$$\mathcal{R}^p(A(\mathsf{M})) = \mathcal{R}^p(\overline{A}(\mathsf{M})) \cup R^{p-1}(\overline{A}(\mathsf{M})) \qquad (8)$$

for $0 \le p \le \ell$. We prove $\mathcal{R}^p(\overline{A}(\mathsf{M})) = \mathcal{R}^p(A(\mathsf{M}))$ by induction. The case $p = 0$ follows from (8). The induction step is obtained by combining (8) with Theorem 3.1. □

3.3 Top and bottom

The two ends of the resonance filtration (5) have straightforward descriptions. First, it will be convenient to have some notation.

Definition 3.8. If π is a partition of $[n]$ with k parts, let P_π denote the codimension-k subspace of $\Bbbk^n$ given by equations

$$\sum_{j \in \pi_i} x_j = 0$$

for $1 \leq i \leq k$. If $k = 1$, we recover $\overline{V} = P_{\{[n]\}}$. At the other extreme, if each block of π is a singleton, then $P_\pi = \{0\}$.

Dually, let Q_π denote the k-dimensional subspace of $\Bbbk^n$ given as the span of vectors

$$\sum_{j \in \pi_i} e_j$$

for $1 \leq i \leq k$. Clearly, P_π and Q_π are complementary subspaces (with respect to the distinguished basis in V.)

Proposition 3.9. *For any matroid* M *on* $[n]$*, let* π *denote the partition of* $[n]$ *given by simplification (Definition 2.9.) Then* $\mathcal{R}^0_1(\mathsf{M}) = P_\pi$*.*

Proof. The simplification map $s\colon \mathsf{M} \to \mathsf{M}_s$ (Definition 2.9) gives an isomorphism of complexes

$$A(s)\colon (A(\mathsf{M}), v) \to (A(\mathsf{M}_s), s(v))$$

for all $v \in V$. For a simple matroid, clearly $\mathcal{R}^0(\mathsf{M}_s) = \{0\}$, so $\mathcal{R}^0(\mathsf{M}) = s^{-1}(0)$, which is the subspace P_π. $\square$

At the other extreme, by the invariance of Euler characteristic, we have

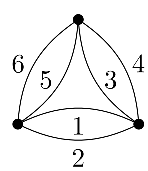
$$\sum_{p=0}^{\ell-1} (-1)^p \dim_{\Bbbk} H^p(\overline{A}(\mathsf{M}), v) = (-1)^{\ell-1}\beta(\mathsf{M}),$$

for any $v \in \overline{V}$, where $\beta(\mathsf{M})$ denotes Crapo's beta invariant. We recall that $\beta(\mathsf{M}) \neq 0$ if and only if M is connected. If $\mathsf{M} = \mathsf{M}(\mathcal{A})$ for an arrangement $\mathcal{A}$, it is usual to say $\mathcal{A}$ is irreducible to mean $\mathsf{M}(\mathcal{A})$ is connected.

If $\mathsf{M}(\mathcal{A})$ is connected, then for any $v \in \overline{V}$, the nonzero Euler characteristic implies that $H^p(\overline{A}(\mathsf{M}), v) \neq 0$ for some p. By Theorem 3.1, $H^{\ell-1}(\overline{A}(\mathsf{M}), v) \neq 0$, which proves the following.

Proposition 3.10 ([35]). *If* M *is connected of rank* ℓ*, then* $\mathcal{R}^{\ell-1}(\overline{A}(\mathsf{M})) = \overline{V} = P_{\{[n]\}}$*.*

Example 3.11. Consider the matroid $\mathsf{M}(G)$, where

$G =$ 

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Using Proposition 3.9, we see $\mathcal{R}^0(G) = P_{12|34|56} = V(x_1 + x_2, x_3 + x_4, x_5 + x_6)$, a 3-dimensional subspace. Since $\mathsf{M}(G)$ is connected of rank 2, $\mathcal{R}^1(G) = R^2(G) = \overline{V} = V(x_1 + \cdots + x_6)$. $\lozenge$

3.4 Upper bounds

Most of the results we present in this paper give lower bounds for the resonance varieties: that is, conditions which imply nonvanishing cohomology. Here, we give two upper bounds. The first uses a result of Schechtman and Varchenko [30] to give a weak but easily stated upper bound.

Theorem 3.12. *If* M *is a matroid of rank* ℓ*, then*

$$\mathcal{R}^p(\mathsf{M}) \subseteq \bigcup_{X \in L^{\mathrm{irr}}_{\leq p+1}(\mathsf{M})} P_{\{X,[n]-X\}}$$

for all $0 \leq p \leq \ell$.

In other words, if $H^p(A(\mathsf{M}), v) \neq 0$, then there exists an irreducible flat X of rank at most $p+1$ for which $v \in P_{\{X,[n]-X\}}$; equivalently, for which $v \in \overline{V}(\mathsf{M})$ and $\sum_{i \in X} v_i = 0$.

Proof. In [30], the authors construct a map of cochain complexes

$$(F(\mathsf{M}), d) \xrightarrow{S(v)} (A(\mathsf{M}), v) \tag{9}$$

for each $v \in V(\mathsf{M})$. The complex $(F(\mathsf{M}), d)$ is isomorphic to the $\Bbbk$-dual of $(A(\mathsf{M}), \partial_A)$; in particular it is exact [30, Cor. 2.8] and does not depend on v.

Let $v(X) := \sum_{i \in X} v_i$. The determinant formula [30, Thm. 3.7] expresses the determinant of $S(v)$ in terms of powers of products of $v(X)$'s. In particular, it follows that $S^p(v)$ is an isomorphism if $v(X) \neq 0$ for all $X \in L^{\mathrm{irr}}_{\leq p}(\mathsf{M})$. Suppose, then, that $v(X) \neq 0$ for all irreducible X of rank at most $p+1$. This implies (9) is an isomorphism up to degree $p+1$, so $H^q(A(\mathsf{M}), v) = 0$ for all $q \leq p$, from which the claim follows. □

Our second result gives a more refined bound for the largest (nontrivial) resonance variety, based on the main result of Cohen, Dimca and Orlik [7]. Since their result applies to the cohomology of local systems on a hyperplane complement, the proof of this bound requires that the matroid have a complex realization.

Definition 3.13. Let us say that a subset of flats $\mathcal{C}$ *covers* M if there is a surjective function $f\colon [n] \to \mathcal{C}$ for which $i \in f(i)$ for all $1 \leq i \leq n$. For a given cover $\mathcal{C}$, let $P_{\mathcal{C}} = \bigcap_{X \in \mathcal{C}} P_{\{X,[n]-X\}}$, a linear subspace of $\overline{V}$.

For each $p \geq 0$, we define a subspace arrangement in $\overline{V}$ using M: let

$$\mathcal{S}^p(\mathsf{M}) = \bigcup_{\mathcal{C}} P_{\mathcal{C}},$$

where the union is over all subsets $\mathcal{C} \subseteq L^{\mathrm{irr}}_{\leq p+1}(\mathsf{M})$ that cover M.

Finally, say a cover $\mathcal{C}$ of M is *essential* if $|X| > 1$ for all $X \in \mathcal{C}$, and let

$$\mathcal{S}^p_{\mathrm{ess}}(\mathsf{M}) = \bigcup_{\mathcal{C}} P_{\mathcal{C}},$$

where the union is over essential covers $\mathcal{C} \subseteq L^{\mathrm{irr}}_{\leq p+1}(\mathsf{M})$.

We note that if M is irreducible of rank ℓ, then $\mathcal{S}^0(\mathsf{M}) \subseteq \cdots \subseteq \mathcal{S}^\ell(\mathsf{M}) = \overline{V}$.

Theorem 3.14. *If M is a complex-realizable matroid, then $\mathcal{R}^p(\mathsf{M}) \subseteq \mathcal{S}^p(\mathsf{M})$, for all $p \geq 0$.*

Proof. Suppose $\mathcal{A}$ is a complex arrangement of rank ℓ and $\mathsf{M} = \mathsf{M}(\mathcal{A})$. The main result of [29] allows us to translate [7, Thm. 1] into the following statement about resonance. That is, for $v \in \overline{V}(\mathsf{M})$, suppose that for some $i \in [n]$, whenever $i \in X \in L^{\mathrm{irr}}_{<\ell}(\mathsf{M})$, we have $v(X) \neq 0$. Then $H^p(\overline{A}(\mathsf{M}), v) = 0$ for all $p \neq \ell - 1$.

If $v \in \mathcal{R}^{\ell-2}(\mathsf{M})$, then, for all $i \in [n]$, there exists some X for which $i \in X$ and $v \in P_{\{X,[n]-X\}}$, since $P_{\{X,[n]-X\}} = \big\{v \in \overline{V} \colon v(X) = 0\big\}$. So

$$\begin{aligned}
\mathcal{R}^{\ell-2}(\mathsf{M}) &\subseteq \bigcap_{i=1}^{n} \bigcup_{\substack{X \in L^{\mathrm{irr}}_{<\ell}: \\ i \in X}} P_{\{X,[n]-X\}}, \\
&= \bigcup_{\mathcal{C}} \bigcap_{X \in \mathcal{C}} P_{\{X,[n]-X\}}, \qquad (10)\\
&= \mathcal{S}^{\ell-2}(M)
\end{aligned}$$

since $\mathcal{C}$ runs over all covers in $L^{\mathrm{irr}}_{\leq \ell-1}$. To obtain the result for $p < \ell - 1$, we consider the truncation $T_{p+1}\mathsf{M}$ of M to rank $p+1$. This matroid is also realizable (by intersecting $\mathcal{A}$ with a generic linear space of codimension $p+1$) and $L_{\leq p}(\mathsf{M}) = L_{\leq p}(T_{p+1}\mathsf{M})$, so it is enough to apply (10) to $T_{p+1}\mathsf{M}$. □

Corollary 3.15. *If W is a component of $\mathcal{R}^p(\mathsf{M})$ for a complex-realizable matroid M and $p \geq 0$, then there exists a cover $\mathcal{C} \subseteq L^{\mathrm{irr}}_{\leq p+1}(\mathsf{M})$ for which $W \subseteq P_{\mathcal{C}}$.*

Corollary 3.16. *If M is a complex-realizable matroid of rank ℓ, then for all $p \geq 0$,*

$$\mathcal{R}^p(\mathsf{M}) \cap (\Bbbk^\times)^n \subseteq \mathcal{S}^p_{\mathrm{ess}}(\mathsf{M}). \qquad (11)$$

Proof. If $X = \{i\}$, then $P_{\{X,[n]-X\}}$ is contained in the coordinate hyperplane $x_i = 0$, so its intersection with $(\Bbbk^\times)^n$ is empty. □

We will see examples in the next section in which the upper bound above is sharp: that is, the containment (11) is an equality.

Question 3.17. The bound given in Theorem 3.14 is, of course, completely combinatorial, although the result we use from [7] is topological in nature. Can the hypothesis that M is complex-realizable be dropped?

4 Matroid operations and resonance

In this section, we systematically examine the behaviour of resonance under several familiar matroid operations. In some cases, apparently nontrivial components of resonance varieties can be obtained from tautological components belonging to other matroids.

4.1 Naturality

In some cases, resonance behaves well under the morphisms of $\mathcal{M}$ from Section §2.3. Combining Theorem 2.6 with (4), we see that if $f\colon \mathsf{M}_1 \to \mathsf{M}_2$ is morphism, then for all $v \in V(\mathsf{M}_1)$, there is a map of complexes

$$A(f)\colon (A(\mathsf{M}_1), v) \to (A(\mathsf{M}_2), f(v)). \tag{12}$$

Here is an important special case.

Proposition 4.1. *If X is a flat of a matroid M, for any $v \in V(\mathsf{M}_X)$, the complex $(A(\mathsf{M}_X), v)$ is a split subcomplex of $(A(\mathsf{M}), v)$.*

Proof. The inclusion $j\colon \mathsf{M}_X \to \mathsf{M}$ and projection $\phi\colon \mathsf{M} \to \mathsf{M}_X$ given by

$$\phi(i) = \begin{cases} i & \text{if } i \in X; \\ 0 & \text{otherwise} \end{cases}$$

are both complete weak maps. Since $\phi \circ j = \mathrm{id}$, the result follows by naturality. □

Lemma 4.2. *Suppose that $f\colon \mathsf{M}_1 \to \mathsf{M}_2$ is a morphism induced by the identity map on underlying sets. If the posets $L_{\leq k}(\mathsf{M}_1) \cong L_{\leq k}(\mathsf{M}_2)$ for some integer $k \geq 1$, then $\mathcal{R}^p(\mathsf{M}_1) \cong \mathcal{R}^p(\mathsf{M}_2)$ for $0 \leq p \leq k-1$.*

Proof. Consider the map $A(f)\colon (A(\mathsf{M}_1), v) \to (A(\mathsf{M}_2), v)$, for some $v \in V(\mathsf{M}_1)$. The hypotheses imply this is an isomorphism in degrees $\leq k$, so $H^p(A(\mathsf{M}_1), v) \cong H^p(A(\mathsf{M}_2), v)$ for all $0 \leq p \leq k-1$. □

We say that a matroid M is k-generic for some $k \geq 0$ if M has no circuits of size k or smaller.

Proposition 4.3. *Suppose M is k-generic. Then $\mathcal{R}^p(\mathsf{M}) = \{0\}$ for all i, $0 \leq p \leq k-1$.*

Proof. Let F_n denote the free matroid on $[n]$ (the underlying matroid of the Boolean arrangement.) Clearly $A(F_n) = E$, the exterior algebra, and it is straightforward to check that $\mathcal{R}^p(E) = \{0\}$ for $0 \leq p \leq n$. The claim then follows from Lemma 4.2. □

Example 4.4 (Uniform matroids). Consider the uniform matroid $\mathsf{U}_{\ell,n}$ of rank ℓ on $[n]$. Assume $n > \ell$. Then $\mathsf{U}_{\ell,n}$ is k-generic for all $k < \ell$, so $\mathcal{R}^p(\mathsf{U}_{\ell,n}) = \{0\}$ for $0 \leq p < \ell - 1$, and $\mathcal{R}^{\ell-1}(\mathsf{U}_{\ell,n}) = \overline{V}$ because $\mathsf{U}_{\ell,n}$ is connected. ♢

4.2 Sums, submatroids and duals

Let us single out three particularly well-behaved constructions. For convenience, if $X \in L(\mathsf{M})$, we abuse notation and identify $V(\mathsf{M}_X)$ with its image as a coordinate subspace in $V(\mathsf{M})$, supported in the coordinates indexed by X.

Theorem 4.5. *Suppose* M *is a matroid of rank* ℓ *on* $[n]$. *Let* X *be any flat of* M, *and* $\mathsf{M}^{\perp}$ *the dual matroid. Then:*

	Construction	*Resonance*
(1)	$\mathsf{M} = \mathsf{M}_1 \oplus \mathsf{M}_2$	$\mathcal{R}^k(\mathsf{M}) = \bigcup_{p+q=k} \mathcal{R}^p(\mathsf{M}_1) \times \mathcal{R}^q(\mathsf{M}_2)$
(2)	*submatroids*	$\mathcal{R}^p(\mathsf{M}_X) \subseteq \mathcal{R}^p(\mathsf{M}) \cap V(\mathsf{M}_X)$ *for all* $p \geq 0$
(3)	*duality*	$\mathcal{R}^{\ell-p}(\mathsf{M}) \cap (\Bbbk^{\times})^n = \mathcal{R}^{n-\ell-p}(\mathsf{M}^{\perp}) \cap (\Bbbk^{\times})^n,$ *for* $p \geq 0$

Proof. (1) is due to Papadima and Suciu [27, Prop. 13.1], since if $\mathsf{M} = \mathsf{M}_1 \oplus \mathsf{M}_2$, then $A(\mathsf{M}) = A(\mathsf{M}_1) \otimes_{\Bbbk} A(\mathsf{M}_2)$. Claim (2) follows directly from Proposition 4.1. Claim (3) appears as Theorem 27 in [10]. □

4.3 Local components

If X is a flat of an arrangement M on $[n]$, we define a partition $\pi(X)$: let $X = X_1 \vee X_2 \cdots \vee X_r$ be the decomposition of X into connected components. Let $\pi(X)$ be the partition given by $i \sim j \Leftrightarrow \{i, j\} \subseteq X_k$, for all i, j, k, and $i \sim i$ if $i \notin X$.

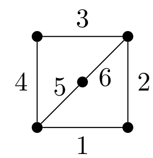
Proposition 4.6. *For any matroid* M, *then for any flat* $X \in L_q(\mathsf{M})$, *there is an equality* $P_{\pi(X)} = V_X \cap R^p(\mathsf{M})$ *for all* p *with* $q - r \leq p \leq \ell$, *where* $r = |\pi(X)|$.

Proof. Since $\mathsf{M}_X = \bigoplus_{i=1}^r \mathsf{M}_{X_i}$, in view of the product formula of Theorem 4.5(1), it is enough to prove the claim when X is connected. In this case, $P_{\pi(X)} = \overline{V}_X$. By Proposition 3.10, $\mathcal{R}^{q-1}(\mathsf{M}_X) = \overline{V}_X$, which is contained in $\mathcal{R}^{q-1}(\mathsf{M}) \cap V_X$ by Theorem 4.5(2). By propagation (Theorem 3.1),

$$\overline{V}_X \subseteq \mathcal{R}^p(\mathsf{M}) \cap V_X \text{ for } q - 1 \leq p \leq \ell. \tag{13}$$

On the other hand, $\mathcal{R}^p(\mathsf{M}) \subseteq \overline{V}$ and $\overline{V} \cap V_X = \overline{V}_X$, so (13) is an equality for all p with $q-1 \leq p \leq \ell$. $\square$

Example 4.7. Let us consider the dual to Example 3.11. That is, $\mathsf{M}^\perp = \mathsf{M}(G^\perp)$, where

$G^\perp =$  $\cdot$

By Proposition 4.3, we see $\mathcal{R}^0(\mathsf{M}^\perp) = \mathcal{R}^1(\mathsf{M}^\perp) = \{0\}$. By Proposition 4.6, each of the circuits of size 4 contribute local components, so each of the 3-dimensional linear spaces $P_{1|2|3456}$, $P_{3|4|1256}$, $P_{5|6|1234}$ are contained in $\mathcal{R}^2(\mathsf{M}^\perp)$.

By Theorem 4.5(3), we see $\mathcal{R}^2(\mathsf{M}^\perp)$ also contains the component $P_{12|34|56}$. With the help of Macaulay2 [21], we find that the upper bound from Theorem 3.14 is sharp, so

$$\mathcal{R}^2(\mathsf{M}^\perp) = \mathcal{S}^2(\mathsf{M}^\perp) = P_{1|2|3456} \cup P_{3|4|1256} \cup P_{5|6|1234} \cup P_{12|34|56}.$$

Since $\mathsf{M}^\perp$ is irreducible of rank 4, we have $\mathcal{R}^3(\mathsf{M}^\perp) = \mathcal{R}^4(\mathsf{M}^\perp) = P_{123456}$, by Proposition 3.9. $\Diamond$

Remark 4.8. As an aside, let's recall an application to the cohomology of the Milnor fibration of a hyperplane arrangement, a full treatment of which can be found in the paper of Papadima and Suciu [28]. The matroid $\mathsf{M}^\perp$ is realized by the arrangement defined by

$$Q = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_1 - x_4)(x_1 - x_5)(x_3 - x_5).$$

Each part of $\pi = 12|34|56$ has even length, so the vector $d = (1,1,1,1,1,1) \in \mathcal{R}^2_{\Bbbk}(\mathsf{M}^\perp)$ if $\operatorname{char} \Bbbk = 2$. From an application of the tangent cone formula (see, for example, [12]), it follows -1 is a monodromy eigenvalue of $H^2(F, \mathbb{C})$, where $F = Q^{-1}(1)$, the Milnor fibre of the arrangement. Our point here is that an apparently nontrivial cohomological phenomenon can result from matroid operations applied to a rather trivial example (Example 3.11). $\Diamond$

4.4 Deletion-contraction

In the case of complex hyperplane arrangements, Cohen [8] showed that there is a long exact sequence relating cohomology of a local system on the complement of an arrangement, its deletion, and its restriction. We will use the combinatorial analogue here in order to examine the behaviour of resonance varieties under deletion and contraction.

If M is a matroid and i_0 is an element which is not a loop, let $\mathsf{M}' := \mathsf{M} - \{i_0\}$ and $\mathsf{M}'' := \mathsf{M}/i_0$ denote the deletion and contraction matroids, respectively. From [24, Thm. 3.65], there is a short exact sequence of $\Bbbk$-modules

$$0 \longrightarrow A(\mathsf{M}') \xrightarrow{j} A(\mathsf{M}) \xrightarrow{\phi} A(\mathsf{M}'')[-1] \longrightarrow 0 \tag{14}$$

where the map ϕ is defined on monomials by

$$\phi(e_{i_0} e_S) = e_S,$$

and $\phi(e_S) = 0$ if $i_0 \notin S$. The inclusion j is a ring homomorphism, induced by the morphism $\mathsf{M}' \hookrightarrow \mathsf{M}$, so (14) is a sequence of $A(\mathsf{M}')$-modules.

For any $v \in \overline{V}(\mathsf{M}')$, then, we obtain a short exact sequence of complexes,

$$0 \longrightarrow (A(\mathsf{M}'), v) \xrightarrow{\quad j \quad} (A(\mathsf{M}), v) \xrightarrow{\quad \phi \quad} (A(\mathsf{M}''), v)[-1] \longrightarrow 0 \tag{15}$$

where we identify $\overline{V}(\mathsf{M}')$ and $\overline{V}(\mathsf{M}'')$ with the coordinate hyperplane $H_{i_0} := P_{i_0|1,\ldots,\hat{i_0},\ldots,n} \subseteq \overline{V}(\mathsf{M})$.

Proposition 4.9. *Let $(\mathsf{M}, \mathsf{M}', \mathsf{M}'')$ be a deletion-contraction triple. Then, for all integers $p \geq 0$ and $k \geq j \geq 0$, we have the following inclusions:*

$$H_{i_0} \cap \mathcal{R}^p_k(\mathsf{M}) - \mathcal{R}^p_{j+1}(\mathsf{M}') \subseteq \mathcal{R}^{p-1}_{k-j}(\mathsf{M}''); \tag{16}$$

$$\mathcal{R}^{p-1}_k(\mathsf{M}'') - \mathcal{R}^p_{j+1}(\mathsf{M}) \subseteq \mathcal{R}^{p+1}_{k-j}(\mathsf{M}'); \tag{17}$$

$$\mathcal{R}^p_k(\mathsf{M}') - \mathcal{R}^{p-2}_{j+1}(\mathsf{M}'') \subseteq \mathcal{R}^p_{k-j}(\mathsf{M}). \tag{18}$$

Proof. Given $v \in H_{i_0}$, consider the long exact sequence in cohomology of (15). By exactness, we have

$$\dim_{\Bbbk} H^p(A(\mathsf{M}), v) \leq \dim_{\Bbbk} H^p(A(\mathsf{M}'), v) + \dim_{\Bbbk} H^{p-1}(A(\mathsf{M}''), v),$$

from which the first containment easily follows. The remaining two are obtained in the same way. ◻

We note that, by putting $j = 0$ and $k = 1$, we can erase the subscripts that keep track of depth.

Corollary 4.10. *Suppose M is k-generic. Then for any element i_0,*

$$\mathcal{R}^p(\mathsf{M} - \{i_0\}) \subseteq \mathcal{R}^p(\mathsf{M}),$$

for all $0 \leq p \leq k$.

Proof. If M is k-generic, the contraction M/i_0 is $(k-1)$-generic. The result then follows by Propositions 4.9(18) and 4.3. ◻

Since all simple matroids are 1-generic, the situation for $p = 1$ is particularly nice. The next example shows that, in general, $\mathcal{R}^p(\mathsf{M}') \not\subseteq \mathcal{R}^p(\mathsf{M})$ for $p > 1$.

Example 4.11. Let $\mathsf{M} = \mathsf{M}(G_1)$ be the matroid of the graph shown in Figure 1. Deleting the horizontal edge gives the graphic matroid of Example 4.7, which we denote for the moment by M'. We saw $W := P_{12|34|56|7} \subseteq \mathcal{R}^2(\mathsf{M}')$: let us parameterize

$$W = \{(-a, a, b, -b, -c, c, 0) \colon a, b, c \in \Bbbk\}\,.$$

If $v \in W$, then $v \notin \mathcal{R}^0(\mathsf{M}'')$ provided either $a + b \neq 0$ or $c \neq 0$. From Proposition 4.9(18), we see $W \subseteq \mathcal{R}^2(\mathsf{M})$. We can repeat this twice to find $W \subseteq \mathcal{R}^2(G_2)$ as well, where G_2 is the graph obtained from K_5 by deleting an edge.

However, the same argument does not allow us to conclude $W \subseteq \mathcal{R}^2(K_5)$: if we contract the bottom edge, we see W is now contained in $\mathcal{R}^0$ of the contraction (Proposition 3.9), so Proposition 4.9 does not apply. Indeed, it turns out that $W \not\subseteq \mathcal{R}^2(K_5)$. Since $W \subseteq \mathcal{S}^2(\mathsf{M}(K_5))$, though, our upper bound is of no use here, and we are forced to verify this with a direct calculation. $\Diamond$

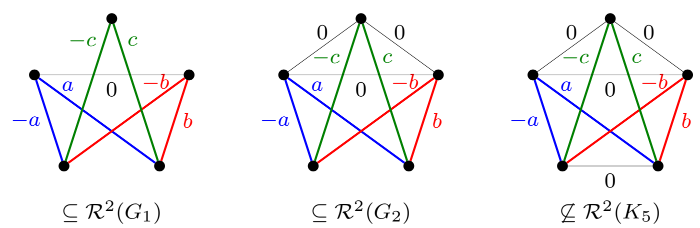


Fig. 1 Adding edges may not preserve resonance (Example 4.11)

Along the same lines, we see also that a component of $\mathcal{R}^2(\mathsf{M})$ contained in a coordinate hyperplane need not be a component of the deletion $\mathcal{R}^2(\mathsf{M}')$.

Example 4.12. Consider the matroid of the graph G from Example 2.7, ordered as shown in (3). With the help of Macaulay 2 and Theorem 3.14, we see

$$W := \{(-a - b, b, 0, a, b, a, -a, -b) \colon a, b \in \Bbbk\}$$

is a component of $\mathcal{R}^2(G)$. Let G' be the graph obtained by deleting the edge with label 0 (see Figure 2). One argues that $W \not\subseteq \mathcal{R}^2(G')$ using Theorem 3.14 as follows. If $v \in W$ and $a, b \neq 0$, we choose an edge incident to a degree vertex, and check that the only irreducible flats X that contain it have $v(X) \neq 0$. It follows that $v \notin \mathcal{S}^2(G')$. $\Diamond$

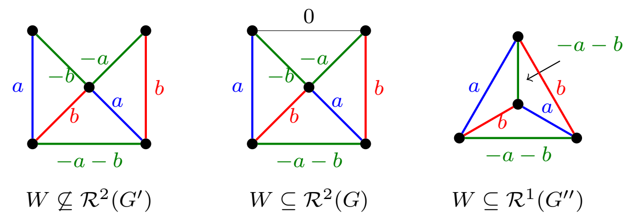


Fig. 2 Deletion may not preserve resonance (Example 4.12)

4.5 Parallel connections

Here, we consider another matroid construction through which resonance varieties can be traced. The underlying data is the following. Suppose M_1 and M_2 are matroids on ground sets S_1 and S_2, respectively, and $X = S_1 \cap S_2$ is a modular flat of M_1 and $(\mathsf{M}_1)_X = (\mathsf{M}_2)_X$. The (generalized) parallel connection $\mathsf{M}_1 \parallel_X \mathsf{M}_2$ is the matroid on $S_1 \cup S_2$ obtained from $\mathsf{M}_1 \oplus \mathsf{M}_2$ by identifying the common copy of X – see [25] for details. Let $\mathsf{M}_{12} = (\mathsf{M}_1)_X = (\mathsf{M}_2)_X$.

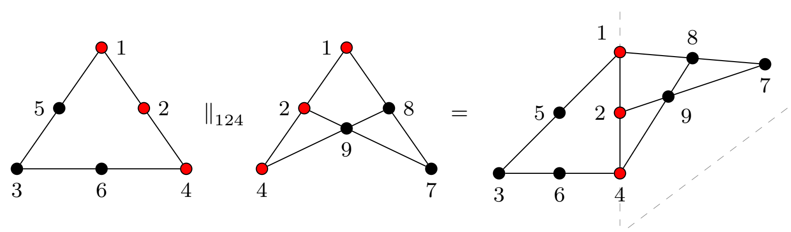


Fig. 3 Parallel connection

The Orlik-Solomon complex of the parallel connection can be described as follows. We begin with the degree-1 part. Let $i_j \colon \mathsf{M}_{12} \to \mathsf{M}_j$ denote the inclusions, for $j = 1, 2$. The identification map $\phi\colon S_1 \sqcup S_2 \to S_1 \cup S_2$. These are all morphisms of $\mathcal{M}$, and they fit in a short exact sequence:

$$0 \longrightarrow V(\mathsf{M}_{12}) \xrightarrow{i_1 \oplus -i_2} V(\mathsf{M}_1) \oplus V(\mathsf{M}_2) \xrightarrow{\phi} V(\mathsf{M}_1 \parallel_X \mathsf{M}_2) \longrightarrow 0$$

By restricting, we also obtain:

$$0 \longrightarrow \overline{V}(\mathsf{M}_{12}) \longrightarrow \overline{V}(\mathsf{M}_1) \oplus V(\mathsf{M}_2) \xrightarrow{\phi} \overline{V}(\mathsf{M}_1 \parallel_X \mathsf{M}_2) \longrightarrow 0 \tag{19}$$

Now let P denote the pushout of graded $\Bbbk$-algebras:

$$\begin{tikzcd} A(\mathsf{M}_{12}) \arrow[r, "A(i_1)"] \arrow[d, "-A(i_2)"'] \arrow[dr, phantom, "\ulcorner"] & A(\mathsf{M}_1) \arrow[d] \\ A(\mathsf{M}_2) \arrow[r] & P \end{tikzcd} \tag{20}$$

We can express the algebra P variously as

$$\begin{aligned} P &\cong \Bbbk \otimes_{A(\mathsf{M}_{12})} \big(A(\mathsf{M}_1) \otimes_{\Bbbk} A(\mathsf{M}_2)\big), \\ &\cong \Bbbk \otimes_{A(\mathsf{M}_{12})} A(\mathsf{M}_1 \oplus \mathsf{M}_2) && \text{by [27, Prop. 13.1];} \\ &\cong A(\mathsf{M}_1 \oplus \mathsf{M}_2) /\!\!/ A(\mathsf{M}_{12}) && \text{see, e.g., [5, p. 349];} \\ &\cong A(\mathsf{M}_1 \parallel_X \mathsf{M}_2). \end{aligned}$$

Remark 4.13. We note that, since X is modular in M_1, $A(\mathsf{M}_1)$ is a free $A(\mathsf{M}_{12})$-module (see [33]). This implies $A(\mathsf{M}_1 \oplus \mathsf{M}_2)$ is also a free $A(\mathsf{M}_{12})$-module. Taking Hilbert series, we obtain

$$h(A(\mathsf{M}_1 \parallel_X \mathsf{M}_2, t) = h(A(\mathsf{M}_1), t) h(A(\mathsf{M}_2), t) / h(A(\mathsf{M}_{12}), t). \tag{21}$$

This amounts to the classical formula relating the characteristic polynomials of the four matroids, so the diagram (20) can be taken as an algebraic refinement. ♢

We now restrict to the classical parallel connection, where X consists of a single element. The description above is particularly straightforward in this case: in particular, the linear map $\overline{\phi}$ in (19) is an isomorphism, and we show next that it induces an algebra isomorphism. In the case of complex hyperplane arrangements, the maps below come from maps of spaces: see [16] and [12, §7]. If $X = \{i\}$ for some $i \in S_1 \cup S_2$, let i' denote its image in a copy of S_2 disjoint from S_1.

Theorem 4.14. *If $X = \{i\}$, there is a short exact sequence*

$$0 \longrightarrow A(\mathsf{M}_1 \parallel_{\{i\}} \mathsf{M}_2)[-1] \longrightarrow A(\mathsf{M}_1 \oplus \mathsf{M}_2) \xrightarrow{A(\phi)} A(\mathsf{M}_1 \parallel_{\{i\}} \mathsf{M}_2) \longrightarrow 0.$$

and an isomorphism $\overline{A}(\mathsf{M}_1) \otimes_{\Bbbk} \overline{A}(\mathsf{M}_2) \cong \overline{A}(\mathsf{M}_1 \parallel_{\{i\}} \mathsf{M}_2)$.

Proof. Since ϕ is surjective, so is $A(\phi)$ (Theorem 2.6), and $A(\mathsf{M}_1 \parallel_{\{i\}} \mathsf{M}_2) \cong A(\mathsf{M}_1 \oplus \mathsf{M}_2)/(r)$, where $r = e_i - e_{i'}$. On the other hand, multiplication by r gives a degree-1 map $A(\mathsf{M}_1 \oplus \mathsf{M}_2)/(r) \to A(\mathsf{M}_1 \oplus \mathsf{M}_2)$. Since r is easily seen to be nonresonant, this map is injective: that is, an isomorphism onto its image, $(r) = \ker A(\phi)$. It follows that the sequence is exact.

To prove the second claim, note the image of the restriction of $A(\phi)$ to $\overline{A}(\mathsf{M}_1)\otimes_{\Bbbk}\overline{A}(\mathsf{M}_2)$ is contained in $\overline{A}(\mathsf{M}_1 \|_{\{i\}} \mathsf{M}_2)$, since the source is generated in degree 1, where the situation is that of (19). Since the target is also generated in degree 1, it follows the map is surjective. To conclude it is an isomorphism, we compare Hilbert series using (21) and (2). □

The effect on resonance varieties is immediate.

Corollary 4.15. *If $X = \{i\}$, the map $\overline{\phi}\colon \overline{V}(\mathsf{M}_1) \oplus \overline{V}(\mathsf{M}_2) \to \overline{V}(\mathsf{M}_1 \|_X \mathsf{M}_2)$ restricts to an isomorphism for each $p \geq 0$:*

$$\mathcal{R}^p(\mathsf{M}_1 \oplus \mathsf{M}_2) \cong \mathcal{R}^p(\mathsf{M}_1 \|_X \mathsf{M}_2).$$

5 Singular subspaces and multinets

So far, we have seen that the resonance varieties of a matroid in top and bottom degrees are easy to account for, and that various resonance components can be obtained from these by comparing with submatroids, duals, deletion and contraction. The lower bounds obtained in this way sometimes match the upper bound given by Theorem 3.14.

Some quite special matroids are known to have additional components in $\mathcal{R}^1(\mathsf{M})$, however. We will assume from now on that $\operatorname{char}\Bbbk = 0$. Building on work of Libgober and Yuzvinsky [22] as well as Falk [17], Falk and Yuzvinsky [19] have characterized these in terms of auxiliary combinatorics. This is the notion of a multinet, and we we briefly recall the construction from [19] in §5.1 with a view to higher-degree generalizations. We refer to Yuzvinsky's survey [36] in particular for a complete introduction.

Some first steps generalizing this theory to $\mathcal{R}^p(\mathsf{M})$ for $p > 1$ appear in [9], in the case of complex hyperplane arrangements, as well as in forthcoming work of Falk [3]. We interpret these constructions in terms of maps of Orlik-Solomon algebras, as in §4.

5.1 $\mathcal{R}^1(\mathsf{M})$: multinets

Definition 5.1. If M is a matroid on $[n]$ and k is an integer with $k \geq 3$, a (k,d)-multinet is a partition $\mathcal{L} := \{\mathcal{L}_1, \ldots, \mathcal{L}_k\}$ of $[n]$ together with a set $\mathcal{X} \subseteq L_2^{\mathrm{irr}}(\mathsf{M})$ with the following four properties:

1. $|\mathcal{L}_s| = d$ for all $1 \leq s \leq k$;
2. If $i, j \in [n]$ belong to different parts of the partition $\mathcal{L}$, then they span a flat in $\mathcal{X}$;
3. For any $X \in \mathcal{X}$, the number $|\mathcal{L}_s \cap X|$ is independent of s;

4. For any $i, j \in \mathcal{L}_s$ for some s, there is a sequence $i = i_0, i_1, \ldots, i_r = j$ for which span $\{i_{q-1}, i_q\} \notin \mathcal{X}$ for all $1 \leq q \leq r$.

The original (equivalent) formulation in [19] replaces M with its simplification M_s and records the number of parallel elements with a multiplicity function. We will say that a simple matroid supports a multinet if it is the simplification of a matroid with a partition as in Definition 5.1.

Theorem 5.2 (Thms. 2.3, 2.4, [19]). *$\mathcal{R}^1_{\Bbbk}(\mathsf{M})$ contains an essential component if and only if M supports a multinet.*

Explicitly, this component is the linear space $Q_{\mathcal{L}} \cap \overline{V}$ (Definition 3.8). Their construction can be interpreted in terms of maps of Orlik-Solomon algebras as follows. First, we note that if $\mathcal{L}$ is a multinet, the partition satisfies the hypothesis of Proposition 2.8, so there is a morphism $p_{\mathcal{L}}\colon \mathsf{M} \to \mathsf{U}_{2,k}$ and a surjection

$$A(p_{\mathcal{L}})\colon \overline{A}(\mathsf{M}) \to \overline{A}(\mathsf{U}_{2,k}).$$

The more interesting aspect is the existence of a right inverse to $A(p_{\mathcal{L}})$. Multinets give the following construction [19].

Proposition 5.3. *If $(\mathcal{L}, \mathcal{X})$ is a multinet on M, there is a ring homomorphism $i_{\mathcal{L}}\colon A(\mathsf{U}_{2,k}) \to A(\mathsf{M})$ defined by*

$$i_{\mathcal{L}}(e_s) = \frac{1}{|\mathcal{L}_s|} \sum_{i \in \mathcal{L}_s} e_i \text{ for all } 1 \leq s \leq k,$$

which restricts to a map $i_{\mathcal{L}}\colon \overline{A}(\mathsf{U}_{2,k}) \to \overline{A}(\mathsf{M})$.

We note that $A(p_{\mathcal{L}}) \circ i_{\mathcal{L}} = \mathrm{id}$. From this it follows that $i_{\mathcal{L}}$ is injective in cohomology. From Example 4.4, $\mathcal{R}^1(\mathsf{U}_{2,k}) = \overline{V}(\mathsf{U}_{2,k})$, and its image in $\mathcal{R}^1(\mathsf{M})$ is just $Q_{\mathcal{L}} \cap \overline{V}$.

Example 5.4 ([19]). Let M be the matroid of the B_3 root system, giving each of the short roots multiplicity two. The corresponding hyperplane arrangement is defined by the polynomial $x^2y^2z^2(x-y)(x+y)(x-z)(x+z)(y-z)(y+z)$. Numbering the points of the matroid $\{1, 1', 2, 2', 3, 3', 4, 5, 6, 7, 8, 9\}$, the dependencies are shown in Figure 4. The multinet $\mathcal{L} = \{11'89|22'67|33'45\}$ has $\mathcal{X} = L_2^{\mathrm{irr}}(\mathsf{M})$, so $Q_{\mathcal{L}} \cap \overline{V}$ is a 2-dimensional, essential component of $\mathcal{R}^1(\mathsf{M})$.

Returning to the upper bound of Corollary 3.16, let $\mathcal{C} = L_2^{\mathrm{irr}}(B_3)$, an essential cover. By direct computation, $P_{\mathcal{C}}$ has dimension 2, so it equals the essential component computed above. ◊

5.2 $\mathcal{R}^{\geq 1}(\mathsf{M})$: singular subspaces

The Multinet Theorem 5.2 does not yet have a complete higher analogue. Here, we indicate some first steps in that direction, beginning with a definition from [9, §3].

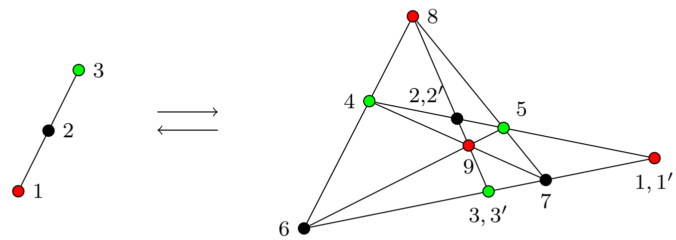


Fig. 4 The $(3,4)$-multinet for the B_3 root system

Definition 5.5. A subspace $W \subseteq \overline{V}(\mathsf{M})$ is called *singular* if the multiplication map $\Lambda^k(W) \to A^k(\mathsf{M})$ is zero, where $k = \dim W$. The *rank* of W is the largest q for which $\Lambda^q(W) \to A^q(\mathsf{M})$ is not the zero map.

Proposition 5.6. *If* $\phi\colon \overline{A}(\mathsf{U}_{q+1,k+1}) \to \overline{A}(\mathsf{M})$ *is a graded homomorphism which is injective in degree* 1*, then the degree*-1 *part of* $\operatorname{im}\phi$ *is a singular subspace of dimension* k *and rank at most* q*.*

Conversely, if W *is a singular subspace of rank* q *in* $\overline{V}(\mathsf{M})$*, there exists a map* ϕ *as above for which* $W = (\operatorname{im}\phi)^1$*.*

Proof. Let $W \subseteq \overline{V}$ be a subspace of dimension k. By inspecting the Orlik-Solomon relations (1), we can identify $\overline{A}(\mathsf{U}_{q+1,k+1})$ with a truncated exterior algebra $\Lambda(W)/(\Lambda^{q+1}(W))$.

If a map $\phi\colon \overline{A}(\mathsf{U}_{q+1,k+1}) \to \overline{A}(\mathsf{M})$ is given, let $W = \phi(\overline{V}(\mathsf{U}_{q+1,k+1}))$. Since $\overline{A}(\mathsf{U}_{q+1,k})^p = 0$ for $p > q$, its image W is singular of rank at most q.

Conversely, the hypothesis implies that the natural map $\Lambda(W) \to A(\mathsf{M})$ factors through $\overline{A}(\mathsf{U}_{q+1,k+1}) \cong \Lambda(W)/(\Lambda^{q+1}(W))$. □

If $\operatorname{char}\Bbbk = 0$, then components of $\mathcal{R}^1(\mathsf{M})$ are just the same as rank-1 singular subspaces, by Theorem 3.5. For higher rank, the situation is more subtle. If W is a singular subspace of rank q and dimension k, the condition implies that the natural homomorphism $\Lambda(W) \to \overline{A}(\mathsf{M})$ factors through $\overline{A}(U_{q+1,k+1})$, a truncated exterior algebra. If the resulting map $\overline{A}(U_{q+1,k+1}) \to \overline{A}(\mathsf{M})$ is injective in cohomology, then $W \subseteq \mathcal{R}^q(\mathsf{M})$.

Example 5.7. The graph G from Example 4.12 provides an interesting example of a singular subspace. $\mathcal{R}^1(G)$ consists of the four local components from the three-element flats. $\mathcal{R}^2(G)$ is more complicated. We find two essential components by first constructing a singular subspace. We label the vertices of G with $\{1,2,\ldots,5\}$ so that edge i has vertices $(i,i+1)$ for $i = 1,2,3$. If edge $i = \{s,t\}$ and $s < t$, let $f_i = x_t - x_s \in \Bbbk[x_1,\ldots,x_5]$. The linear forms $\{f_i \colon 1 \le i \le 8\}$ define the graphic arrangement $\mathcal{A}(G)$ with matroid $\mathsf{M}(G)$.

Following the approach of Cohen et al. [9], we observe that there is a linear relation of cubic polynomials

$$f_1 f_7 f_8 + f_2 f_5 f_8 + f_3 f_5 f_6 = f_4 f_6 f_7.$$

This implies that the polynomial mapping $\Phi\colon U(\mathcal{A}(G)) \subseteq \mathbb{P}^4 \to \mathbb{P}^2$ given by

$$\Phi(x) = [f_1 f_7 f_8 : f_2 f_5 f_8 : f_3 f_5 f_6]$$

has its image inside the projective complement of the arrangement consisting of the three coordinate hyperplanes together with the projective hyperplane orthogonal to $[1:1:1]$. The existence of the induced map in cohomology $\Phi^*\colon \overline{A}(\mathsf{U}_{3,4}) \to \overline{A}(G)$ shows that W is a rank-2 singular subspace, by Proposition 5.6, where

$$W = \left\{ v \in \Bbbk^8 \colon v = (a, b, c, d, b + c, c + d, d + a, a + b) \text{ and } a + b + c + d = 0 \right\}.$$

Below, we will show Φ^* is split, which implies that $H^2(\Phi^*)$ is injective, and $W \subseteq \mathcal{R}^2(G)$.

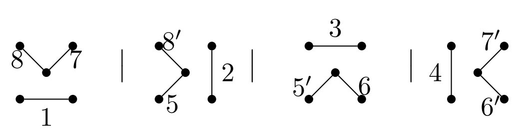
We proceed indirectly to show that W is maximal (i.e., a component.) First, using the cover $\mathcal{C} = L_2^{\mathrm{irr}}(\mathsf{M}) = \{156, 267, 378, 458\}$, we obtain a 3-dimensional linear space

$$P_{\mathcal{C}} = \left\{ v \in \Bbbk^8 \colon v = (c + d, d + a, a + b, b + c, a, b, c, d) \text{ and } a + b + c + d = 0 \right\}.$$

Note $P_{\mathcal{C}}$ is not maximal in $\mathcal{S}^2(G)$: for example, if $\mathcal{C}' = \{1234, 156, 378\}$, $P_{\mathcal{C}} \subsetneq P_{\mathcal{C}'}$. Up to symmetry, though, this is the only subspace in $\mathcal{S}^2(G)$ that properly contains $P_{\mathcal{C}}$, and it has dimension 4. By checking a single $v \in P_{\mathcal{C}'} - P_{\mathcal{C}}$, we see $P_{\mathcal{C}'} \not\subseteq \mathcal{R}^2(G)$.

Now we note that the matroid $\mathsf{M}(G)$ is self-dual, and we may identify $\mathsf{M}(G)$ with its dual via the permutation $\sigma = [56784123]$. Since W is essential, $\sigma(W) \subset \mathcal{R}^2(G)$ as well, by Theorem 4.5(3). However, $\sigma(W) = P_{\mathcal{C}}$, so $P_{\mathcal{C}} \subseteq \mathcal{R}^2(G)$ as well. It follows that W and $\sigma(W)$ are both (maximal) linear components of $\mathcal{R}^2(G)$.

In order to try to imitate the multinet construction (Theorem 5.2), we give the inner four edges of G multiplicity 2, and denote the (non-simple) matroid by $\widetilde{\mathsf{M}}$. Let $\mathcal{L}$ be the partition of $\{1, 2, 3, 4, 5, 5', 6, 6', 7, 7', 8, 8'\}$ given by

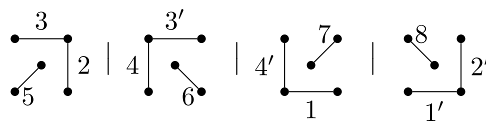
$\mathcal{L} =$ 

Then $W \cong Q_{\mathcal{L}} \cap \overline{V}(\widetilde{\mathsf{M}})$.

The map $A(\mathsf{U}_{3,4}) \to A(\widetilde{\mathsf{M}})$ given by sending the ith generator to the sum of elements in the ith block of $\mathcal{L}$, for $1 \leq i \leq 4$, restricts to Φ^* above. To construct a left-inverse, recall that in Example 2.7 we found a morphism $\underline{f}\colon \mathsf{M} \to \mathsf{U}_{3,4}$ in $\overline{\mathcal{M}}$. A simple check shows that $-A(f) \circ \Phi^*$ is the identity on $\overline{A}(\mathsf{U}_{34})$, so Φ^* is indeed split. ◊

We continue this example by observing that not every essential component of $\mathcal{R}^2(G)$ is a singular subspace.

Example 5.8 (Example 5.7, continued). The second component $P_{\mathcal{C}} = \sigma(W)$ can also be expressed in terms of a partition: $\sigma(W) \cong Q_{\mathcal{L}^*} \cap \overline{V}(\widetilde{\mathsf{M}})$ for another non-simple matroid $\widetilde{M}$, where

$\mathcal{L}^* =$ 

.

The image of $\Lambda^3(\sigma(W))$ in $\overline{A}(G)$ is nonzero, so this subalgebra generated by $\sigma(W)$ does not factor through $\overline{A}(\mathsf{U}_{3,4})$. ◊

However, the partitions $\mathcal{L}$ and $\mathcal{L}^*$ above look qualitatively rather similar. One might hope, then, that Theorem 5.2 admits a combinatorial generalization that treats both components above equally.

Question 5.9. By the theory of multinets, every component of $\mathcal{R}^1(\mathsf{M})$ comes from the tautological resonance of a rank-2 matroid, via a split surjection of Orlik-Solomon algebras. Does every component of $\mathcal{R}^p(\mathsf{M})$ come from a matroid of rank $p+1$, for all $p \geq 1$?

Our last example shows that we cannot always find a uniform matroid with this property for $p = 2$, unlike for $p = 1$; however, since the only simple matroids of rank 2 are uniform, this should not necessarily be seen evidence that the answer is negative.

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