Combinatorics of Theta Bases of Cluster Algebras

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Jan 22–24, 2016

What is a Cluster Algebra?

Cluster algebra is a subalgebra of the rational function field $\mathbb{Q}(x_1, ..., x_n)$

generated by a distinguished set of elements (cluster variables)

grouped into overlapping n-element subsets (clusters)

defined by a recursive procedure (mutation).

Example: (skew-symmetric) cluster algebras of Rank 2

Definition

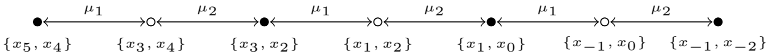
Let $r \in \mathbb{Z}_{>0}$.

- the cluster variables $\{x_m : m \in \mathbb{Z}\}$ are defined by exchange relation:

$$x_{m-1}x_{m+1} = x_m^r + 1$$

- the cluster algebra $\mathcal{A}(r,r)$ is the subring of $\mathbb{Q}(x_1, x_2)$ generated by cluster variables
- the sets $\{x_m, x_{m+1}\}$ for $m \in \mathbb{Z}$ are called clusters
- $x_m^i x_{m+1}^j$ with $i, j \geq 0$ is called a cluster monomial

Mutation:



Example: (skew-symmetric) cluster algebras of Rank 2

Example ($r = 1$)

The are five cluster variables:

$x_1,\, x_2,\, x_3 = \frac{x_2+1}{x_1},\, x_4 = \frac{x_1+x_2+1}{x_1x_2},\, x_5 = \frac{x_1+1}{x_2}.$

$(\dots, x_{-1} = x_4,\, x_0 = x_5;\, x_6 = x_1, \quad x_7 = x_2, \dots)$

Example ($r = 2$)

The first few cluster variables:

$x_0 = \frac{x_1^2+1}{x_2},\, x_{-1} = \frac{x_1^4+x_2^2+2x_1^2+1}{x_1x_2^2},\, x_{-2} = \frac{x_1^6+x_2^4+2x_1^2x_2^2+3x_1^2+2x_2^2+3x_1^2+1}{x_1^2x_2^3}$

$x_3 = \frac{x_2^2+1}{x_1},\ \ x_4 \ = \frac{x_2^4+x_1^2+2x_2^2+1}{x_1^2x_2},\ \ x_5 \ = \frac{x_2^6+x_1^4+2x_1^2x_2^2+3x_2^2+2x_1^2+3x_2^2+1}{x_1^3x_2^2}$

Structure Theorems

Theorem (S. Fomin, A. Zelevinsky, 2002)

All elements in a cluster algebra are Laurent polynomials.

Theorem (S. Fomin, A. Zelevinsky, 2003)

Finite type cluster algebras (i.e. those with finitely many clusters) are classified using finite type Dynkin diagrams.

Theorem (Irelli, Keller, Labardini-Fragoso,Plamondon, 2012)

The cluster monomials are linearly independent in a skew-symmetric cluster algebra.

Theorem (Lee-Schiffler, 2013; Gross-Hacking-Keel-Kontsevich, 2014)

The Laurent polynomial of each cluster variable has only nonnegative coefficients.

Theorem (Gross-Hacking-Keel-Kontsevich, 2014)

Any cluster algebra has an additive basis which includes the cluster monomials and is strongly positive.

A list of bases of cluster algebras

Name of Basis	Cluster algebras	Authors	Source of definition
Cluster monomials	Dynkin type	Caldero-Keller	Cluster category
Atomic	Type A and $\tilde{A}$	Sherman-Zelevinsky, Irelli, Dupont-Thomas	Algebra (Generators of the cone of positive elements)
Bracelet	From marked surfaces	Musiker-Schiffler-Williams	Geometry (surfaces with boundary and marked points)
Standard monomial	(Quantum) Acyclic	Berenstein-Fomin-Zelevinsky	Algebra (Some Monomials of cluster variables)
Triangular	(Quantum) Acyclic	Berenstein-Zelevinsky	Algebra (Bar-invariance)
Dual canonical	(Quantum) Acyclic	Nakajima,Kimura-Qin	Representation theory, Geometry (Nakajima's quiver varieties)
Bangles	From marked surfaces	Musiker-Schiffler-Williams	Geometry (Surfaces with boundary and marked points)
Generic	(Quantum) Acyclic	Dupont, Plamondon	Representation theory
Dual semicanonical	Attached to unipotent cells in Lie groups	Geiss-Leclerc-Schröer	Representation theory
Greedy	(Quantum) Rank 2	Lee-Li-Rupel-Zelevinsky	Algebra and Combinatorics
Theta function	mid($\mathcal{A}_{\mathrm{prin}}$)	Gross-Hacking-Keel-Kontsevich	Geometry (Tropical geometry, cluster varieties)

Globally Compatible Collections (GCCs)

In [Lee,Li,Mills, 2015], we constructed a new basis $\tilde{x}[\mathbf{a}]$:

Let Q be the quiver. B be the signed adjacency matrix ($b_{ij} = 1$ if there is an arrow $i \to j$, $b_{ij} = -1$ if there is an arrow $j \to i$ ...)
Given $\mathbf{a} = (a_i) \in \{0,1\}^n$, a GCC is an n-tuple $\mathbf{s} = (s_1, , \dots, s_n) \in \{0,1\}^n$ such that $0 \le s_i \le a_i$ and $(s_i, a_j - s_j) \neq (1,1)$ for every $(i,j) \in Q$.
Define

$$\tilde{x}[\mathbf{a}] := \left(\prod_{i=1}^{n} x_i^{-a_i}\right) \sum_{\mathbf{s}} \prod_{i=1}^{n} x_i^{\sum_{j=1}^{n}(a_j - s_j)[b_{ij}]_+ + s_j[-b_{ij}]_+}.$$

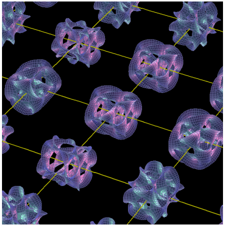
Extend the definition multiplicatively $\tilde{x}[\mathbf{a}]\tilde{x}[\mathbf{a}'] = \tilde{x}[\mathbf{a} + \mathbf{a}']$ for certain $\mathbf{a}, \mathbf{a}'$.

Theorem

Let $\mathcal{U}$ be the upper cluster algebra of a (not necessarily acyclic) cluster algebra $\mathcal{A}$ of geometric type. Then $\tilde{x}[\mathbf{a}] \in \mathcal{U}$ for all $\mathbf{a} \in \mathbb{Z}^n$.

Theorem

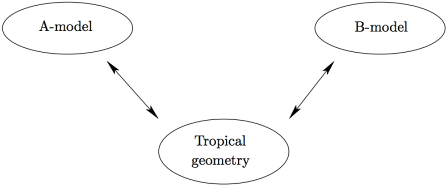
Let $\mathcal{A}$ be an acyclic cluster algebra of geometric type, and fix an acyclic seed. Then $\{\tilde{x}[\mathbf{a}]\}_{\mathbf{a} \in \mathbb{Z}^n}$ form a basis of $\mathcal{A}$.



String theory

At any point in 4-dimensional space time there is a hidden 6-dimensional Calabi-Yau manifold.

Mirror symmetry refers to a situation when two Calabi-Yau manifolds look very different but are equivalent in string theory. It involves a relationship between symplectic geometry (A-model) and complex geometry (B-model).



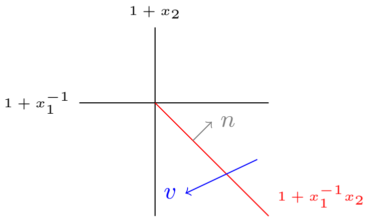
Theta function bases and broken lines

Consider the geometry in $\mathbb{R}^2$ (rank 2 case):

- A wall is a pair (d, f_d), where d is a line or ray , f_d is a formal Laurent series.
- A scattering diagram is a collection of walls.
- Given a wall (d, f_d) and a direction $v \in M$ transversal to d, we define an automorphism

$$p_{v,d}(\mathbf{x}^m) = \mathbf{x}^m f_d^{m \cdot n}$$

where n is the primitive normal vector of d satisfying $v \cdot n < 0$.



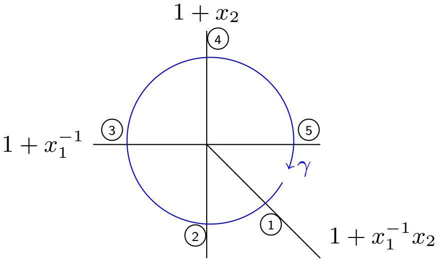
$$\begin{aligned}
&p_{v,d}(x_1^{m_1} x_2^{m_2}) \\
&= x_1^{m_1} x_2^{m_2} (1 + x_1^{-1} x_2)^{(m_1,m_2)\cdot(1,1)} \\
&= x_1^{m_1} x_2^{m_2} (1 + x_1^{-1} x_2)^{m_1+m_2}
\end{aligned}$$

Theta function bases and broken lines

Given a path $\gamma(t)$ passing through walls (d_1, f_{d_1}), (d_2, f_{d_2}),..., (d_k, f_{d_k}) for $t = t_1, t_2, \ldots, t_k$. we define an automorphism

$$p_\gamma = p_{\gamma'(t_k),d_k} \circ \cdots \circ p_{\gamma'(t_1),d_1}$$

A scattering diagram is consistent if p_γ =identity, for every closed path γ.



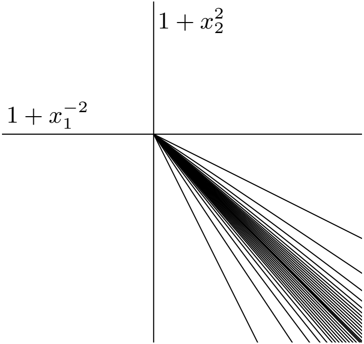
Verification:

$$x_1 \xrightarrow{①} x_1 + x_2 \xrightarrow{②} x_1 + x_2 + x_1x_2 \xrightarrow{③} x_1(1 + x_2) \xrightarrow{④} x_1 \xrightarrow{⑤} x_1$$

$$x_2 \xrightarrow{①} \frac{x_2(x_1+x_2)}{x_1} \xrightarrow{②} \frac{x_2^2}{x_1(1+x_2)} \xrightarrow{③} \frac{x_1x_2(1+x_2)}{1+x_1+x_1x_2} \xrightarrow{④} \frac{x_1x_2}{1+x_1} \xrightarrow{⑤} x_2$$

Theta function bases and broken lines

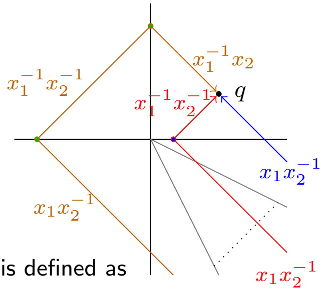
In general, the scattering diagram can be very complicated:



Theta function bases and broken lines

A broken line $\gamma : (-\infty, 0] \to \mathbb{R}^2$ with initial exponent m_0 and endpoint q is:

- a piecewise linear path which may bend when it crosses walls, and
- a monomial $c(\ell)\mathbf{x}^{m(\ell)}$ on each domain of linearity ℓ.



The theta function corresponding to m and Q is defined as

$$\theta_{Q,m} = \sum_{\gamma} \mathrm{Mono}(\gamma)$$

where γ runs through broken lines with initial exponent m and end point Q, and $\mathrm{Mono}(\gamma)$ the monomial attached to the last domain of linearity of γ.

Example

In the picture, $m = x_1x_2^{-1}$, $\theta_{Q,m} = x_1x_2^{-1} + x_1^{-1}x_2 + x_1^{-1}x_2^{-1} = \dfrac{x_1^2 + x_2^2 + 1}{x_1x_2}$.

Theta function bases and broken lines

Theorem (Gross, Hacking, Keel, Kontsevich *arXiv*, 2014)

Theta functions form a basis of an acyclic cluster algebra.

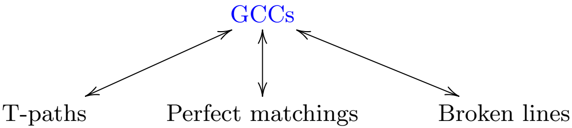
(Moreover they proved the positivity conjecture and the Fock-Goncharov conjecture)

Theorem (Cheung, Gross, Muller, Musiker, Rupel, Stella, Williams, *arXiv*, 2015)

For a rank 2 cluster algebra, the greedy basis coincides with the theta function basis.

Theorem (Lee, L–, Nguyen, in preparation)

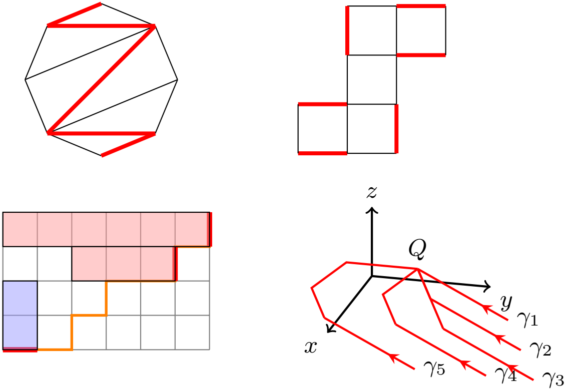
For a type-A cluster algebra, we give explicit bijections from the set of GCCs to the set of T-paths, the set of perfect matchings, and the set of broken lines, respectively.



Theta function bases and broken lines

Combinatorial models for cluster variables in type A:

- Triangularization and T-paths
- Snake diagram and perfect matchings
- Maximal Dyck path and compatible pairs (GCCs)
- Scattering diagram and broken lines



The bijection {GCCs} $\xleftrightarrow{1:1}$ {Broken lines}

The idea of the proof:

Step 1. For each GCC, construct a sequence of walls $(w_1, \dots, w_\ell)$ where the broken line γ bends.

Step 2. Give a combinatorial description of the direction vectors of the broken line in all the domains of linearity.

Step 3. Show that the above sequence of walls and the direction vectors uniquely determine a valid broken line.

Step 4. Use [GHKK] to conclude that all broken lines are obtained this way.

The bijection $\{\text{GCCs}\} \xleftrightarrow{1:1} \{\text{Broken lines}\}$: Step 1.

Consider the quiver Q: $1 \to 2 \leftarrow 3 \leftarrow 4$.
Compute the cluster variable with denominator $x_1x_2x_3x_4$.
A GCC is obtained by fill cells with 0 or 1 and avoid $\boxed{1} \to \boxed{0}$.

$$\boxed{} \to \boxed{} \leftarrow \boxed{} \leftarrow \boxed{}$$

There are 6 GCCs.

$$\boxed{0} \to \boxed{0} \leftarrow \boxed{0} \leftarrow \boxed{0}$$
$$\boxed{0} \to \boxed{1} \leftarrow \boxed{0} \leftarrow \boxed{0}$$
$$\boxed{1} \to \boxed{1} \leftarrow \boxed{0} \leftarrow \boxed{0}$$
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Compute the cluster variable with denominator $x_1x_2x_3x_4$.
A GCC is obtained by fill cells with 0 or 1 and avoid $\boxed{1} \to \boxed{0}$.

$$\boxed{} \to \boxed{} \leftarrow \boxed{} \leftarrow \boxed{}$$

There are 6 GCCs. Label an arrow $i \to j$ by the rule $\boxed{0} \xrightarrow{x_i} \boxed{0}$, $\boxed{1} \xrightarrow{x_j} \boxed{1}$.

$$\begin{array}{ccccccccc}
\boxed{0} & \xrightarrow{x_1} & \boxed{0} & \xleftarrow{x_3} & \boxed{0} & \xleftarrow{x_4} & \boxed{0} & \qquad & x_1x_3x_4 \\
\boxed{0} & \to & \boxed{1} & \leftarrow & \boxed{0} & \xleftarrow{x_4} & \boxed{0} & & x_4 \\
\boxed{1} & \to & \boxed{1} & \xleftarrow{x_2} & \boxed{0} & \leftarrow & \boxed{0} & & x_2 \\
\boxed{0} & \xrightarrow{x_2} & \boxed{1} & \leftarrow & \boxed{1} & \xleftarrow{x_4} & \boxed{0} & & x_2x_4 \\
\boxed{1} & \xrightarrow{x_2} & \boxed{1} & \xleftarrow{x_2} & \boxed{1} & \leftarrow & \boxed{0} & & x_2^2 \\
\boxed{1} & \xrightarrow{x_2} & \boxed{1} & \xleftarrow{x_2} & \boxed{1} & \xleftarrow{x_3} & \boxed{1} & & x_2^2x_3
\end{array}$$

The corresponding cluster variable is obtained:

$$\frac{x_1x_3x_4 + x_4 + x_2 + x_2x_4 + x_2^2 + x_2^2x_3}{x_1x_2x_3x_4}$$

The bijection $\{\text{GCCs}\} \xleftrightarrow{1:1} \{\text{Broken lines}\}$: Step 1.

Consider the quiver Q: $1 \to 2 \leftarrow 3 \leftarrow 4$.
Compute the cluster variable $x[1,1,1,1]$ with denominator $x_1x_2x_3x_4$.
A GCC is obtained by fill cells with 0 or 1 and avoid $\boxed{1} \to \boxed{0}$.

$$\boxed{} \to \boxed{} \leftarrow \boxed{} \leftarrow \boxed{}$$

There are 6 GCCs. The broken line bends where the cell is filled with 0:

$\boxed{0} \to \boxed{0} \leftarrow \boxed{0} \leftarrow \boxed{0}$	Walls 4,3,1,2
$\boxed{0} \to \boxed{1} \leftarrow \boxed{0} \leftarrow \boxed{0}$	Walls 4,3,1
$\boxed{1} \to \boxed{1} \leftarrow \boxed{0} \leftarrow \boxed{0}$	Walls 4,3
$\boxed{0} \to \boxed{1} \leftarrow \boxed{1} \leftarrow \boxed{0}$	Walls 4,1
$\boxed{1} \to \boxed{1} \leftarrow \boxed{1} \leftarrow \boxed{0}$	Walls 4
$\boxed{1} \to \boxed{1} \leftarrow \boxed{1} \leftarrow \boxed{1}$	Not bend

where Wall i is the hyperplane $x_i = 0$.

The bijection $\{\text{GCCs}\} \overset{1:1}{\longleftrightarrow} \{\text{Broken lines}\}$: Step 2.

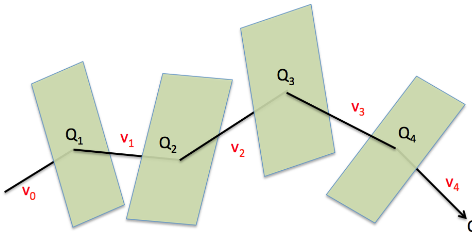
Take $Q = (q_1, q_2, q_3, q_4)$ in general position and $0 < q_1 \ll q_2 \ll q_3 \ll q_4$.

$$\boxed{0} \rightarrow \boxed{0} \leftarrow \boxed{0} \leftarrow \boxed{0} \qquad \text{Walls 4,3,1,2}$$

The direction vectors for the domains of linearity are:

$$\mathbf{v}_0 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Each coordinate is in $\{-1, 0, 1\}$, and can be read from the GCC.



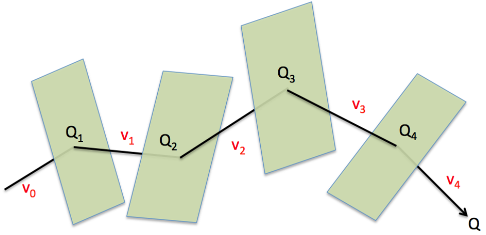
The bijection $\{\text{GCCs}\} \overset{1:1}{\longleftrightarrow} \{\text{Broken lines}\}$: Step 3.

Let Q_i be the i-th vertex of the broken line (and denote $Q_5 = Q$).

$$Q_4 = Q_5 - q_2\mathbf{v_4} = \begin{bmatrix} q_1 \\ 0 \\ q_3 \\ q_4 \end{bmatrix}, \qquad Q_3 = Q_4 - q_1\mathbf{v_3} = \begin{bmatrix} 0 \\ -q_1 \\ q_3 - q_1 \\ q_4 \end{bmatrix},$$

$$Q_2 = Q_3 - (q_3 - q_1)\mathbf{v_2} = \begin{bmatrix} -q_3 + q_1 \\ -q_1 \\ 0 \\ q_4 \end{bmatrix}, \quad Q_1 = Q_2 - q_4\mathbf{v_1} = \begin{bmatrix} -q_4 - q_3 + q_1 \\ q_4 - q_1 \\ -q_4 \\ 0 \end{bmatrix}.$$

Thus $Q_{i+1} - Q_i \in \mathbb{R}^+\mathbf{v}_i$.



The bijection $\{\text{GCCs}\} \xleftrightarrow{1:1} \{\text{Broken lines}\}$: Step 4.

It follows from [GHKK] that, for $m_0 = -\mathbf{v}_0$, and a general Q, the cluster variable is equal to the theta function:

$$x[1,1,1,1] = \theta_{Q,m_0}$$

Since we constructed the correct number of broken lines, the mapping

$$\{\text{GCCs}\} \longrightarrow \{\text{Broken lines}\}$$

is a bijection. $\square$

Thank you!