Combinatorial Algebra meets Algebraic Combinatorics
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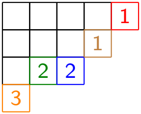
Partial Maps on Littlewood-Richardson Tableaux

Markus Schmidmeier (Florida Atlantic University)



$$g : \boxed{3} \mapsto \boxed{2} \mapsto \boxed{1}\ ,\ \boxed{2} \mapsto \boxed{1}$$

$$\tilde{g} : \boxed{3} \mapsto \boxed{2} \mapsto \boxed{1}\ ,\ \boxed{2} \mapsto \boxed{1}$$

A report on a joint project with
Justyna Kosakowska (Nicolaus Copernicus University)

I. Littlewood-Richardson tableaux

Definition:

An LR-tableau of shape (α, β, γ) is a Young diagram of shape β in which the region $\beta \setminus \gamma$ contains α'_1 entries $\boxed{1}$, ..., α'_s entries $\boxed{s}$, where $s = \alpha_1$ is the length of α', such that

- in each row, the entries are weakly increasing,
- in each column, the entries are strictly increasing,
- the lattice permutation property holds: For each $\ell > 1$ and each column c: on the right hand side of c, the number of entries $\ell - 1$ is at least the number of entries ℓ.

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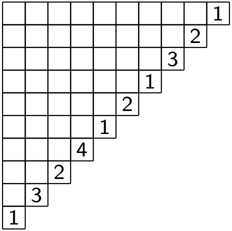
Example:



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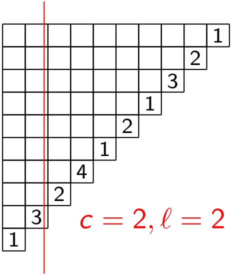


$c = 2, \ell = 2: \quad \#\{\boxed{1}\,'s\} \geq \#\{\boxed{2}\,'s\}$

LR-coefficients in algebra

LR-tableaux occur in many exciting situations in algebra, but on the surface it appears that only their *number* is needed:

The LR-coefficient $c_{\alpha,\gamma}^{\beta}$ counts the number of LR-tableaux of shape (α, β, γ).

- Symmetric functions: Product of Schur polynomials $s_\alpha \cdot s_\gamma = \sum_\beta c_{\alpha,\gamma}^{\beta}\, s_\beta$
- Horn's Problem: There are Hermetian matrices A, B, C with eigenvalues α, β, γ and $A + C = B$ if and only if $c_{\alpha,\gamma}^{\beta} \neq 0$
- Green-Klein Theorem: There is a short exact sequence of finite abelian p-groups (or of nilpotent linear operators) $0 \to N_\alpha \to N_\beta \to N_\gamma \to 0$ if and only if $c_{\alpha,\gamma}^{\beta} \neq 0$

Recall:

$$N_\alpha = \bigoplus_{i=1}^{s} \mathbb{Z}/(p^{\alpha_i}) \quad \text{or} \quad N_\alpha = \bigoplus_{i=1}^{s} k[T]/(T^{\alpha_i}) \quad \text{if} \quad \alpha = (\alpha_1, \ldots, \alpha_s)$$

The tableau of an embedding

Let $0 \to N_\alpha \xrightarrow{f} N_\beta \to N_\gamma \to 0$ be a short exact sequence. Often we will just consider the monomorphism $f : N_\alpha \to N_\beta$, or the embedding $(A \subset B)$ where $A = \operatorname{Im} f$ and $B = N_\beta$.

Suppose in an embedding $(A \subset B)$, the module A has Loewy length r (so r is minimal with $p^r A = 0$). Consider the modules

$$B/A, \quad B/pA, \quad \ldots, \quad B/p^r A = B$$

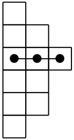
and their corresponding partitions

$$\gamma = \gamma^0, \quad \gamma^1, \quad \ldots, \quad \gamma^r = \beta.$$

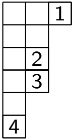
Definition: The tableau of the embedding $(A \subset B)$ is given by the Young diagram β where in each skew diagram $\gamma^i \setminus \gamma^{i-1}$ the boxes are labelled by $\boxed{i}$.

Example: For the embedding $((p^2, p, 1)) \subset \frac{\mathbb{Z}}{(p^6)} \oplus \frac{\mathbb{Z}}{(p^4)} \oplus \frac{\mathbb{Z}}{(p)}$, the above modules have partitions $(5,2), (5,2,1), (5,3,1), (5,4,1), (6,4,1)$.

P :



Γ :



The Green-Klein Theorem revisited

Hence the Green-Klein Theorem really is the following statement:

Theorem (Green, Klein): Let α, β, γ be partitions.

- If $0 \to N_\alpha \to N_\beta \to N_\gamma \to 0$ is a short exact sequence with tableau Γ, then Γ is a Littlewood-Richardson tableau of shape (α, β, γ).
- Conversely, for each Littlewood-Richardson tableau Γ of shape (α, β, γ), there exists a short exact sequence $0 \to N_\alpha \to N_\beta \to N_\gamma \to 0$ with tableau Γ.

For abelian p-groups, the Hall polynomial counts the embeddings corresponding to Γ, it is a monic polynomial of degree $n_\beta - n_\alpha - n_\gamma$.

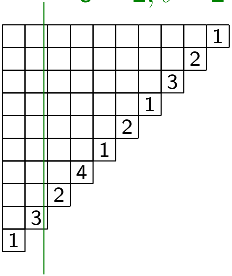
For k-linear operators, k an algebraically closed field, the set of embeddings $f : N_\alpha \to N_\beta$ with cokernel N_γ forms a variety, with irreducible components indexed by the LR-tableaux.

II. The lattice permutation property revisited

Recall that the lattice permutation property (LPP) states that in the LR-tableau Γ, for each $\ell > 1$ and each column c: on the right hand side of c, the number of entries $\ell - 1$ is at least the number of entries ℓ.

Example:

$c = 2, \ell = 2: \quad \#\{\boxed{1}\,'s\} \geq \#\{\boxed{2}\,'s\}$



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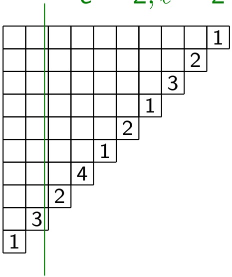

Equivalent to (LPP): For each $\ell > 1$ and each row $r > 1$: the number of entries $\ell - 1$ in row $r - 1$ or above is at least the number of entries ℓ in row r or above.

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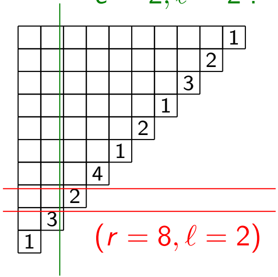


$\#\{\boxed{1}\text{'s in rows } \leq 7\} \geq$

$\#\{\boxed{2}\text{'s in rows } \leq 8\}$

Equivalent to (LPP): For each $\ell > 1$ and each row $r > 1$: the number of entries $\ell - 1$ in row $r - 1$ or above is at least the number of entries ℓ in row r or above.

Klein tableaux

Definition: Let Γ be an LR-tableau. A Klein tableau refining $\Gamma$is a map f which assigns to each box b with entry $e > 1$ the row of a corresponding box with entry $e - 1$ such that

1. if a box b occurs in the m-th row, then $f(b) < m$,
2. if a box b with entry $e > 1$ lies in the m-th row, and the box above has entry $e - 1$ then $f(b) = m - 1$,
3. the number of boxes b with entry $e > 1$ such that $f(b) = r$ is at most the number of boxes in row r with entry $e - 1$, and
4. in each row, for each entry $e > 1$, the map f is weakly increasing.

Notation: We indicate the map f by adding to each box b with entry $e > 1$ as subscript the row of $f(b)$.

Remark: We have just seen that each LR-tableau can be refined to a Klein tableau (by adding subscripts).

Abelian groups with a p^2-bounded subgroup

Theorem (Hunter-Richman-Walker '69, Kosakowska-S '15): Let α, β, γ be partitions such that all parts of α are at most 2. We consider short exact sequences $\mathcal{E} : 0 \to N_\alpha \to N_\beta \to N_\gamma \to 0$. There are one-to-one correspondences:

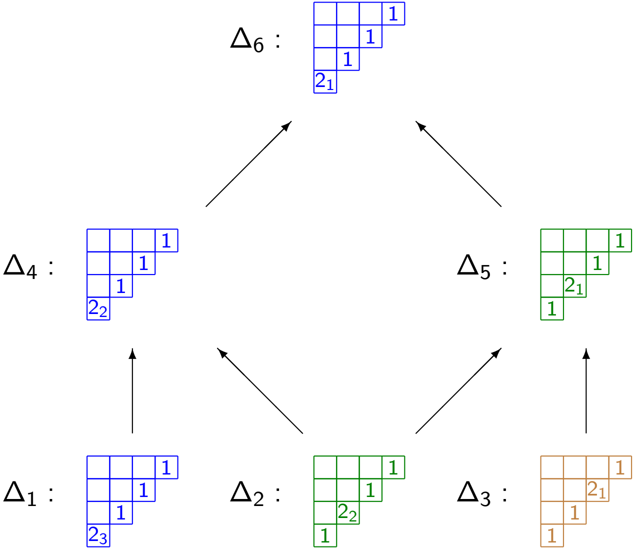
$$\{\text{Klein tableaux of shape } (\alpha, \beta, \gamma)\}$$
$$\stackrel{1-1}{\longleftrightarrow} \quad \{\text{short exact sequences } \mathcal{E} \text{ of abelian } p\text{-groups}\}/\cong$$
$$\stackrel{1-1}{\longleftrightarrow} \quad \{\text{short exact sequences } \mathcal{E} \text{ of } T\text{-invariant subspaces}\}/\cong$$

Note: If all parts of α are at most 1, then Klein tableaux are just LR-tableaux. (For given (α, β, γ), there is at most one: $c^{\beta}_{\alpha,\gamma} \leq 1$.)

Note: If α has parts 3, then a combinatorial classification of the isomorphism types of sequences $\mathcal{E}$ may not be possible.

Example: For $\alpha = (2, 1, 1)$, $\beta = (4, 3, 2, 1)$, $\gamma = (3, 2, 1)$, there are the following three LR-tableaux and six Klein tableaux.

The example in more detail:



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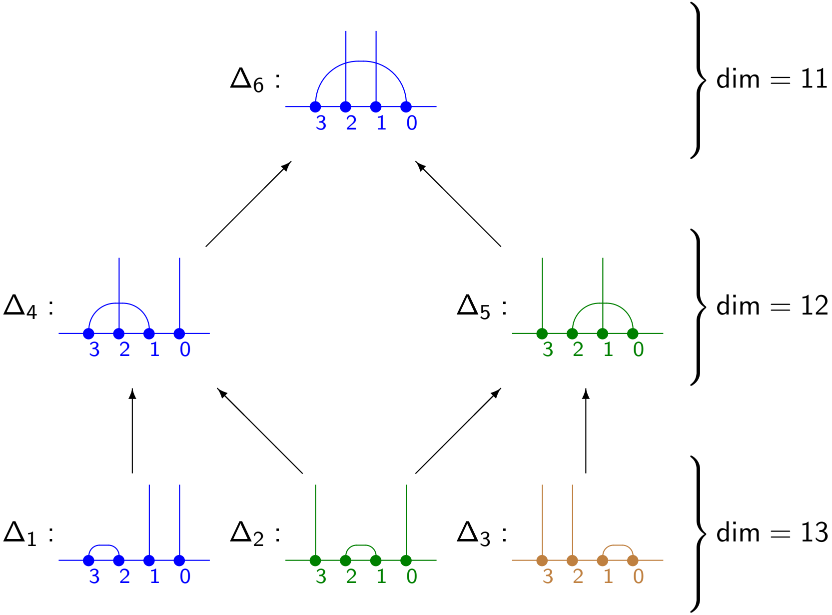

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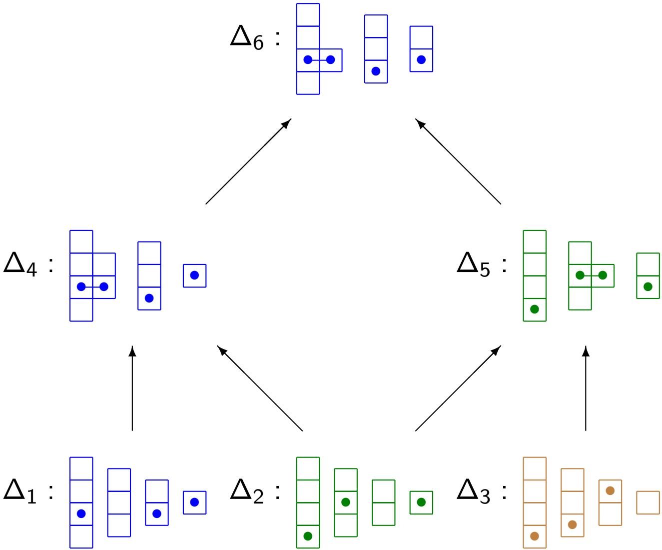


III. Partial maps

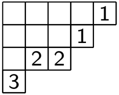
Definition: A partial map g on an LR-tableau Γ assigns to each box $\boxed{e}$ with entry $e > 1$ a box with entry $e - 1$ such that

1. g is one-to-one,
2. for each box b, the row of $g(b)$ is above the row of b, and
3. if the box b has entry e, and the box b' above it has entry $e - 1$ then $g(b) = b'$.

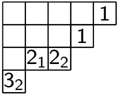
Definition: Let Γ be an LR-tableau. We say two partial maps g, g' are equivalent if $g' = \pi^{-1} g \pi$ holds for some permutation π of the boxes in Γ which preserves entries and rows.

Example: For $\alpha = (3, 2)$, $\beta = (4, 3, 3, 1)$, $\gamma = (3, 2, 1)$, there is one LR-tableau, one Klein tableau and two equivalence classes of partial maps.

Γ:



Π :



$g : \boxed{3} \mapsto \boxed{2_1} \mapsto \boxed{1}\,, \boxed{2_2} \mapsto \boxed{1}$

$\tilde{g} : \boxed{3} \mapsto \boxed{2_2} \mapsto \boxed{1}\,, \boxed{2_1} \mapsto \boxed{1}$

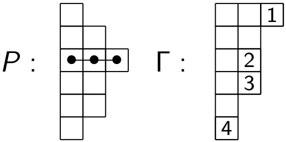
Modules with a cyclic submodule

Definition: An indecomposable embedding $(A \subset B)$ is a pole if A is cyclic.

Properties of the tableau of a pole: Suppose t is the Loewy length of A.

- In Γ, each entry $\boxed{1}, \dots, \boxed{t}$ occurs exactly once.
- The sequence of rows for $\boxed{1}, \dots, \boxed{t}$, is strictly increasing.
- Hence in each column, the entries are subsequent numbers.

Theorem (Kaplansky): Each pole is determined uniquely, up to isomorphy, by the strictly increasing sequence of rows in which the entries $\boxed{1}, \dots, \boxed{t}$ occur.



Direct sums of poles

Note that the tableau of a pole admits a unique partial map; this map has exactly one orbit.

Definition: A partial map g on a tableau Γ has the empty box property (EBP) if for each row r there are at least as many columns in Γ of exactly $r-1$ empty boxes, as there are jumps in row r.

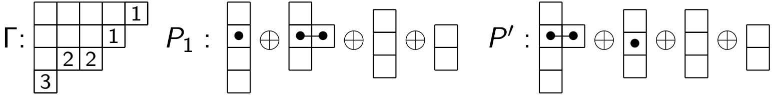
Theorem (Kosakowska-S '15): For an LR-tableau Γ, there is a one-to-one correspondence:

$$\{\text{partial maps on } \Gamma \text{ with (EBP)}\} \stackrel{1-1}{\longleftrightarrow} \{\text{direct sums of poles with tableau } \Gamma\}/\cong$$

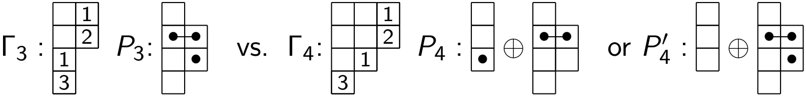
Remark: The previous theorem is the special case where $\alpha_1 \leq 2$.

Some examples...

The previous example:

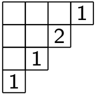


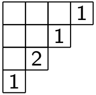
Nonexample and example:

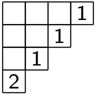


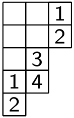
IV. The box relation

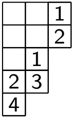
Definition: Suppose that two LR-tableaux $\Gamma, \widetilde{\Gamma}$ have the same shape, both admit a partial map with (EBP). The tableaux are in box relation, $\widetilde{\Gamma} <_{\text{box}} \Gamma$, if $\widetilde{\Gamma}$ is obtained from Γ by exchanging the entries in two columns such that the smaller entries are in the lower position in $\widetilde{\Gamma}$.

Example: 

$<_{\text{box}}$ 

$<_{\text{box}}$ 

Question: 

$\overset{?}{<}_{\text{box}}$ 

Suppose that $\widetilde{\Gamma} <_{\text{box}} \Gamma$ are two LR-tableaux in box relation and that k is an algebraically closed field. We can construct a family of embeddings M_λ, $\lambda \in k$, such that M_λ has tableau $\widetilde{\Gamma}$ if $\lambda \neq 0$ and M_0 has tableau Γ.

Boundary order for LR-tableaux

Definition: Two LR-tableaux $\Gamma, \widetilde{\Gamma}$ of the same shape are in boundary relation, $\widetilde{\Gamma} \leq_{\text{boundary}} \Gamma$, if the following condition is satisfied.

$$\mathbb{V}_{\Gamma} \cap \overline{\mathbb{V}}_{\widetilde{\Gamma}} \neq \emptyset$$

We obtain as a consequence:

Proposition: Suppose $\Gamma, \widetilde{\Gamma}$ are tableaux of the same shape. Then:

$$\widetilde{\Gamma} <_{\text{box}} \Gamma \quad \Longrightarrow \quad \widetilde{\Gamma} \leq_{\text{boundary}} \Gamma \quad \Longrightarrow \quad \widetilde{\Gamma} \leq_{\text{deg}} \Gamma.$$

Here, $\leq_{\text{deg}}$ is the usual degeneration relation.

Theorem (Kosakowska-S-Thomas '14): Suppose that α, β, γ are partitions such that $\beta \setminus \gamma$ is a horizontal and vertical strip. Then the partial orders $\leq^*_{\text{box}}, \leq^*_{\text{boundary}}, \leq_{\text{deg}}$ are equivalent on the set of LR-tableaux of shape (α, β, γ).

V. Summary

- The positivity of the LR-coefficient decides about the existence of short exact sequences $0 \to N_\alpha \to N_\beta \to N_\gamma \to 0$ of finite abelian p-groups, or of nilpotent linear operators.
- Each such sequence corresponds to an LR-tableau; in the operator case, the varieties $\overline{\mathbb{V}}_\Gamma$ form the irreducible components of the representation space $\mathbb{V}^{\beta}_{\alpha,\gamma}$.
- If $\alpha_1 \leq 2$ then the Klein tableaux correspond to the isomorphism types of the short exact sequences. Their number (group case) and their geometric behavior (operator case) can be read off from the arc diagram.
- More general, partial maps on LR-tableaux with (EBP) determine the isomorphism types of direct sums of poles. Box moves give rise to one-parameter families of s.e.s.
- If $\beta \setminus \gamma$ is a horizontal and vertical strip, then the boundary of the irreducible components of the representation space is understood combinatorially.

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Thank You!