

Factorisations of a group element, Hurwitz action and shellability

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joint work with **Henri Mühle** (École Polytechnique, France)

Outline

- 1 Framework and example:
generated group, Hurwitz action, factorisations, shellability
- 2 Motivations:
noncrossing partition lattices of reflection groups
- 3 Some results and a conjecture:
compatible order on the generators, Hurwitz-transitivity, shellability

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Generated group and reduced decompositions

- (G, A) generated group
- $A \subseteq G$ generates G as a monoid
- Let $g \in G$. Write $g = a_1 a_2 \dots a_n$, with $a_i \in A$.
Length of g : $\ell_A(g) :=$ minimal such n .

Reduced decompositions of g

$\text{Red}_A(g) := \{(a_1, \dots, a_n) \mid a_i \in A, g = a_1 \dots a_n\}$, where $n = \ell_A(g)$.

Example. $G = S_4$ $A = T := \{\text{all transpositions } (ij)\}$.

$g = (1\ 2\ 3\ 4)$ $\ell_T(g) = 3$ Reduced decompositions of g :

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Hurwitz action

Hurwitz moves

Fix $g \in G$. Take $(a_1, \dots, a_n) \in \text{Red}_A(g)$. For $1 \leq i \leq n-1$ define:

$$\begin{aligned}\sigma_i \cdot (a_1, \dots, a_{i-1}, & \quad a_i \quad , \quad a_{i+1} \quad , a_{i+2}, \dots, a_n) \\ = & (a_1, \dots, a_{i-1}, \quad a_i a_{i+1} a_i^{-1} \quad , \quad a_i \quad , a_{i+2}, \dots, a_n)\end{aligned}$$

Assumption: For any $(a_1, \dots, a_n) \in \text{Red}_A(g)$ and any $1 \leq i \leq n-1$, $a_i a_{i+1} a_i^{-1}$ and $a_{i+1}^{-1} a_i a_{i+1} \in A$. (e.g., A stable by conjugacy)

This defines an action on $\text{Red}_A(g)$ by the braid group B_n [Hurwitz action].

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1 \rangle_{\text{grp}}$$

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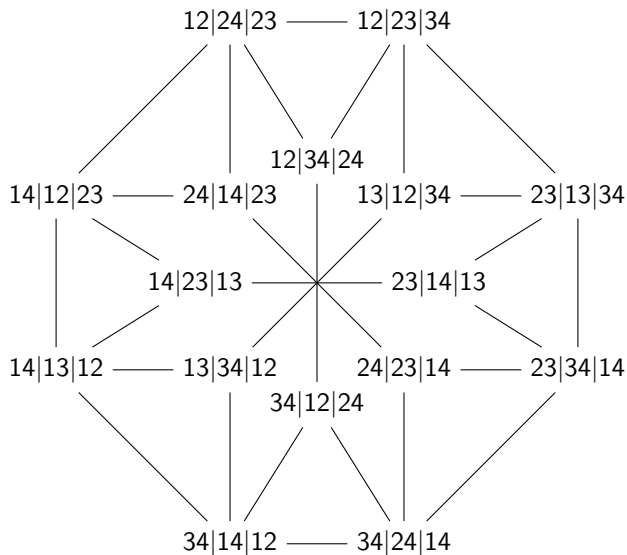
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Example: Hurwitz graph of $\text{Red}_T((1\ 2\ 3\ 4))$



Factorisation poset

Prefix order

Equip G with a partial order \leq_A :

$$\begin{aligned}x \leq_A y &\Leftrightarrow x \text{ is a } \mathbf{prefix} \text{ of a reduced decomposition of } y \\ &\Leftrightarrow \ell_A(x) + \ell_A(x^{-1}y) = \ell_A(y)\end{aligned}$$

Factorisation poset of g

$$[e, g]_A := \{x \in G \mid x \leq_A g\}$$

- $[e, g]_A$ is a graded poset (by ℓ_A);
- Hasse diagram of the poset $[e, g]_A$ corresponds to geodesics from e to g in the Cayley graph of (G, A) ;
- for $x, y \in [e, g]_A$: $x \leq_A y$ if and only if a reduced decomposition of x is a subword of a reduced decomposition of y ; [by assumption on conjugacy-stability]

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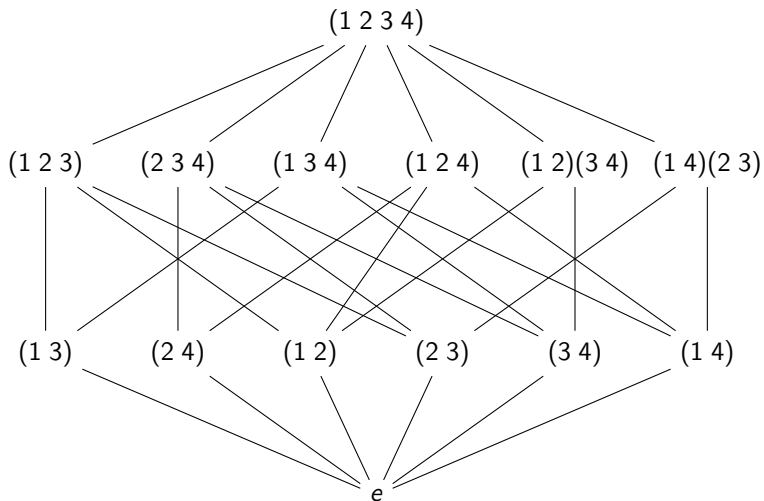
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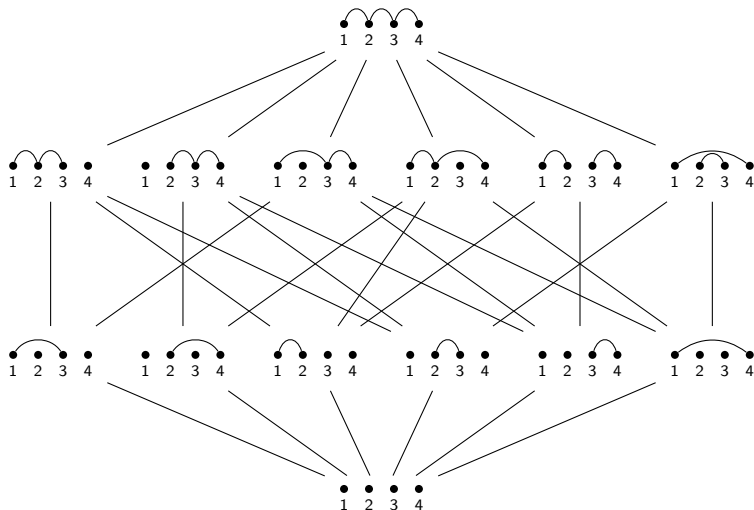
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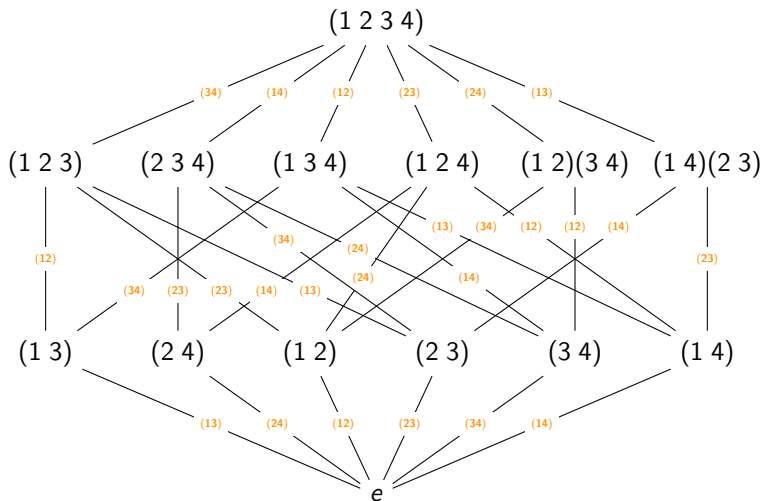
Example: $[e, (1\ 2\ 3\ 4)]_T$ in (S_4, T)



$[e, (1\ 2\ 3\ 4)]_T$ in $(S_4, T) \simeq$ Noncrossing partitions



Example: $[e, (1\ 2\ 3\ 4)]_T$ in (S_4, T)



Notes: $\{\text{maximal chains of } [e, g]_A\} \longleftrightarrow \text{Red}_A(g)$

$\forall x \leq_A y, [x, y]_A \simeq [e, x^{-1}y]_A$

Shellability

Definition

A graded poset P is **shellable** if its *order complex* is shellable, i.e.: there is a total order on the maximal chains $C_1 \prec \cdots \prec C_r$ such that $\forall i < j, \exists k < j$ with

- $C_i \cap C_j \subseteq C_k \cap C_j$; and
- C_k and C_j differ by only one element.

Interest: P shellable \Rightarrow the order complex is Cohen-Macaulay, homotopy-equivalent to a wedge of spheres...

\leadsto **General question 2** : Is $[e, g]_A$ shellable?

Combinatorial criterion: P **EL-shellable** $\Rightarrow P$ shellable. [Björner-Wachs]

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P is EL-shellable if there exists a labelling of the edges (by a totally ordered set) such that for any interval $I \subseteq P$, there is a unique *increasing* maximal chain of I , and this is the lex. smallest among all maximal chains.

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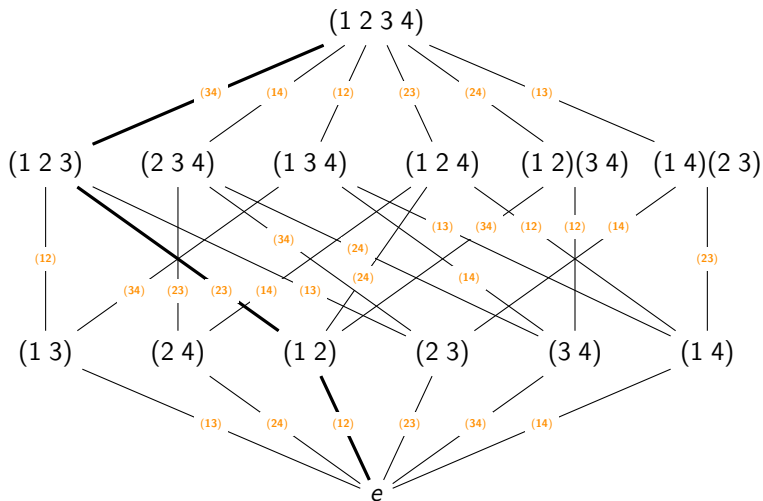
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$(12) \prec (13) \prec (14) \prec (23) \prec (24) \prec (34)$

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Motivation

- W .. finite Coxeter group, or well-generated complex reflection group
- T .. set of all reflections of W
- c .. Coxeter element of W
- W -noncrossing partitions: interval $[e, c]_T$ in (W, \leq_T) $\sim NC_W(c)$

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Theorem (Deligne, 1974; Bessis & Corran, 2006; Bessis, 2006)

For any well-generated complex reflection group W , and any Coxeter element $c \in W$, the braid group $B_{\ell_T(c)}$ acts transitively on $Red_T(c)$.

- Uniform proof only for Coxeter groups!
- Useful property to construct a presentation of the dual braid monoid and of the braid group of W [Bessis]

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Theorem (Björner & Edelman, 1980; Reiner, 1997; Athanasiadis, Brady & Watt, 2007; Mühle, 2015)

For any well-generated complex reflection group W , and any Coxeter element $c \in W$, the poset $NC_W(c)$ is shellable.

- Uniform proof only for Coxeter groups! [ABW]

The Goal

- present a framework to relate
 - ▶ transitivity of the Hurwitz action on $\text{Red}_A(g)$ (General Question 1)
 - ▶ shellability of $[e, g]_A$ (General Question 2)
- help answering these questions by checking “simple” local criteria
- apply this to interesting examples

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Chain-connectedness

Definition

P graded poset. Define the **chain graph** of P to be the graph with vertices the maximal chains of P , and C connected to C' whenever they differ by only one element.

Say P is **chain-connected** if the chain graph is connected.

Observations:

- P shellable $\Rightarrow P$ chain-connected
- Hurwitz-transitivity on $\text{Red}_A(g) \Rightarrow [e, g]_A$ chain-connected

Proposition

Assume

- $[e, g]_A$ is **chain-connected**; and
- for all $x \in [e, g]_A$, with $\ell_A(x) = 2$, the Hurwitz action of B_2 on $\text{Red}_A(x)$ is transitive (**local Hurwitz transitivity**)

Then the Hurwitz action is **transitive** on $\text{Red}_A(g)$.

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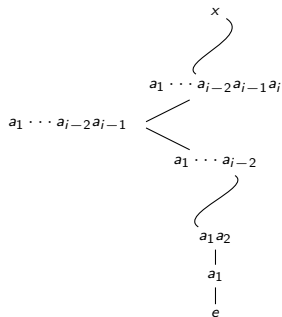
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Hurwitz action on the maximal chains

Hurwitz action corresponds to “taking detours”

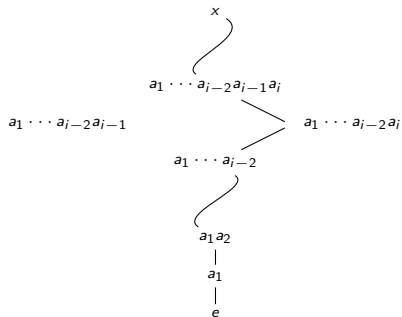
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Compatible generator orders

- G, A, g as before
- assume from now on that $\text{Red}_A(g)$ is finite
- $A_g := \{a \in A \mid a \leq_A g\}$ generators below g .

Definition (Mühle & R, 2015)

A total order \prec on A_g is **g -compatible** if for any $x \leq_A g$ with $\ell_A(x) = 2$, there exists a unique $(s, t) \in \text{Red}_A(x)$ with $s \preceq t$.

- inspired by definition of c -compatible reflection order for Coxeter groups [Athanasiadis, Brady & Watt, 2007], but forgetting the geometry;
- gives EL-shellability in rank 2.

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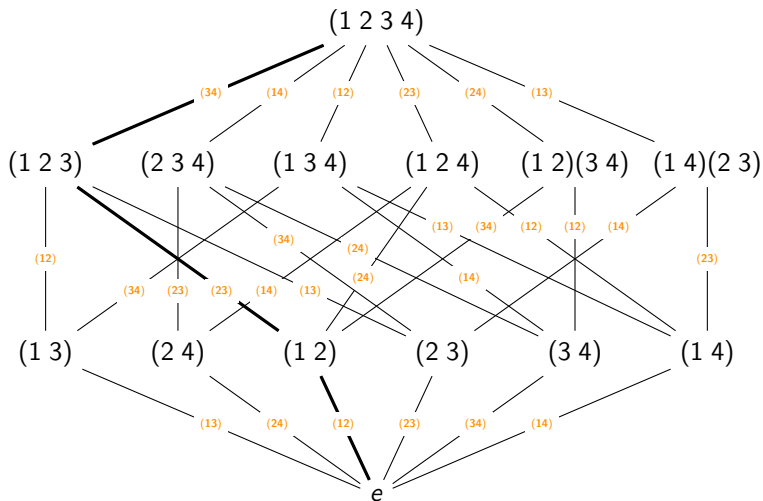
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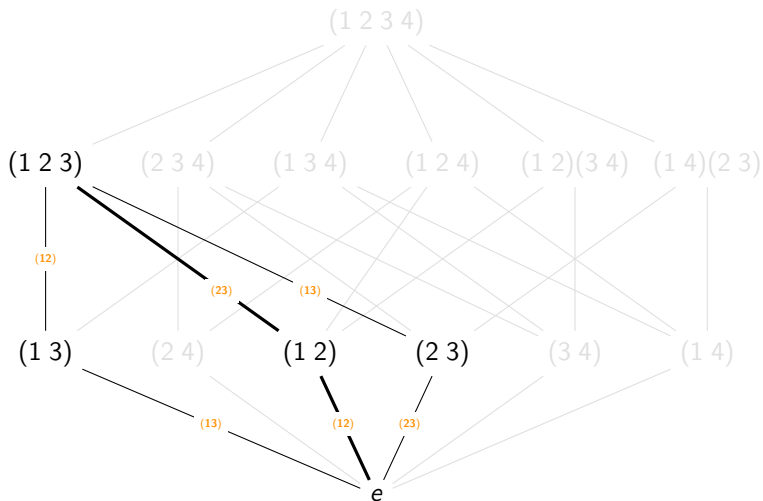
- inspired by definition of c -compatible reflection order for Coxeter groups [Athanasiadis, Brady & Watt, 2007], but forgetting the geometry;
- gives EL-shellability in rank 2.

Example: $[e, (1\ 2\ 3\ 4)]_T$ in (S_4, T)



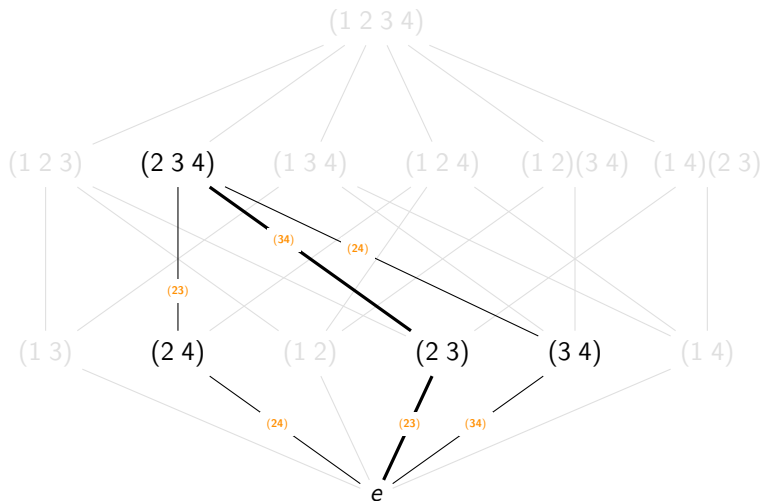
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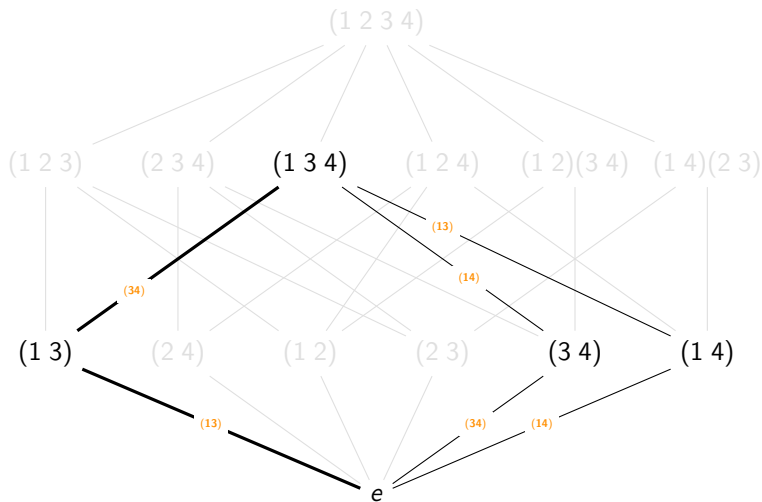
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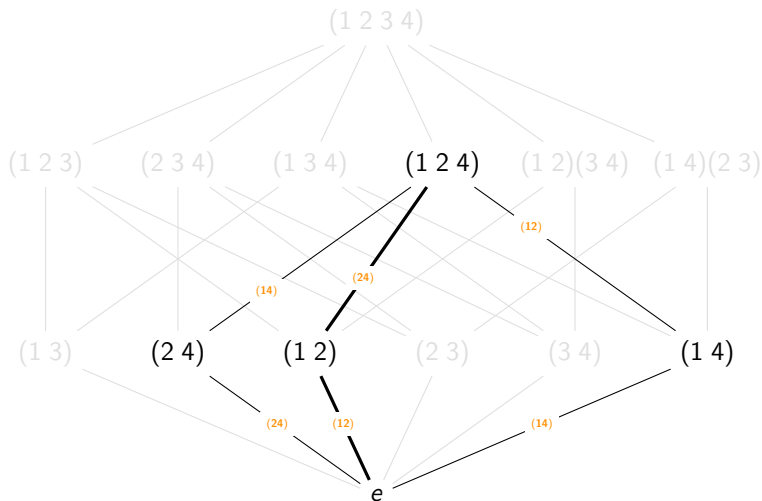
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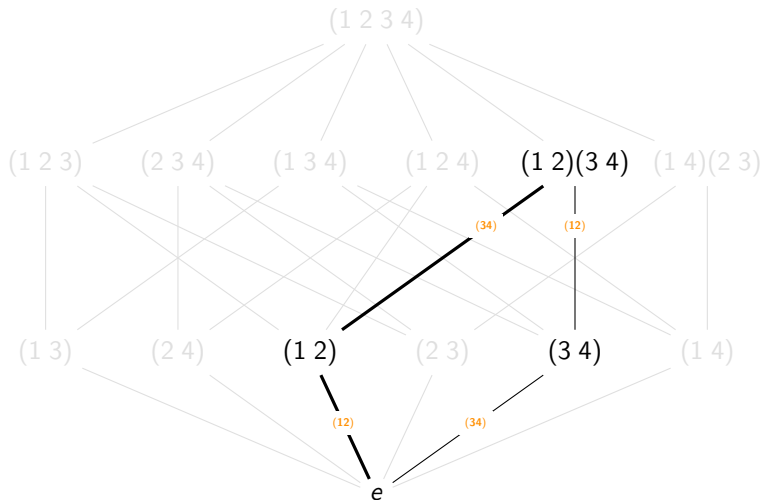
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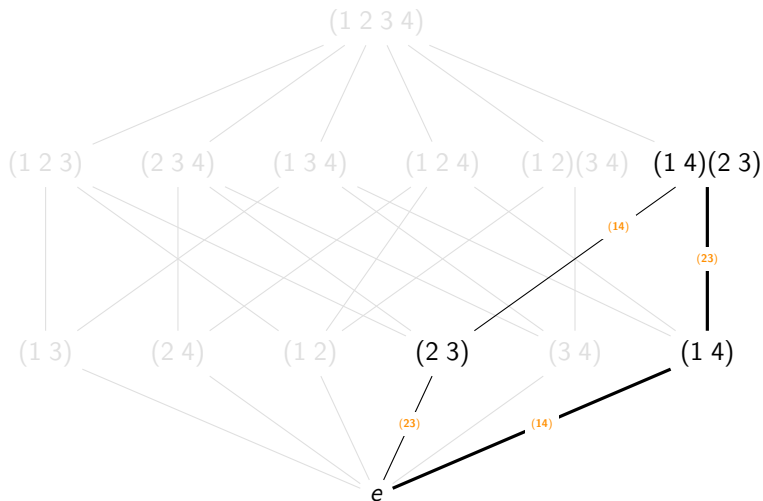
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Compatible orders and Hurwitz transitivity

Proposition (Rank 2 case)

Suppose $\ell_A(g) = 2$. Then:

\exists a g -compatible order on $A_g \iff$

the Hurwitz action of B_2 on $\text{Red}_A(g)$ is transitive.

Corollary (arbitrary rank)

\exists a g -compatible order on $A_g \implies$ local Hurwitz transitivity
(i.e., for all $x \in [e, g]_A$ with $\ell_A(x) = 2$, the Hurwitz action of B_2 on $\text{Red}_A(x)$ is transitive).

- the converse is false.
- Consequence of corollary:
 \exists compatible order + chain-connectedness \implies Hurwitz transitivity.
- Note: \exists compatible order $\not\Rightarrow$ Hurwitz transitivity.

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Proof:

- In rank 2, any Hurwitz orbit has the form $g = a_1 a_2 = a_2 a_3 = \cdots = a_{s-1} a_s = a_s a_1$.
- Assume there is no rising decomposition, then $a_1 \prec a_s \prec a_{s-1} \prec \cdots \prec a_3 \prec a_2 \prec a_1$, impossible.
- so at least one rising decomposition for each orbit. □

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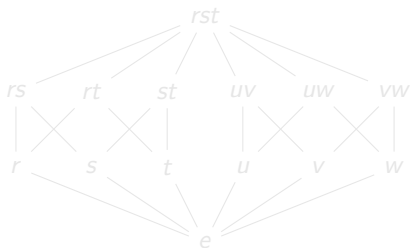
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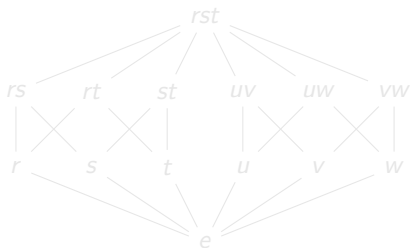
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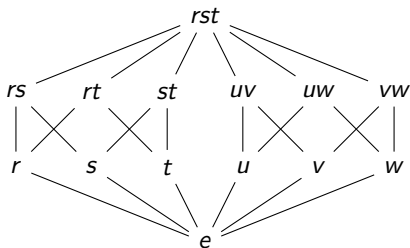
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Compatible orders and shellability

Conjecture (Mühle & R, 2015)

Let G , A , g be as before. Suppose

- there exists a *g -compatible order* on A_g ;
- any interval of $[e, g]_A$ is *chain-connected*.

Then $[e, g]_A$ is *(EL)-shellable*.

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We reduced the conjecture to:

Conjecture (Mühle & R, 2015)

Same hypotheses.

Then for any generator a in A_g (excepted the \prec -smallest one), there exists another generator b in A_g such that

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Further questions

- Applications to specific groups:
 - ▶ complex reflection groups (need to construct uniformly a compatible order!);
 - ▶ (generalized) alternating groups;
 - ▶ (generalized) braid groups
 - ▶ $GL_n(\mathbb{F}_q)$ [Huang-Lewis-Reiner]
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