Splendid Complexes


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Motivation

Let X be a smooth projective toric variety with the polynomial ring S its Cox ring.

GEOMETRY VERSUS ALGEBRA:

- A coherent sheaf on X has a locally free resolution of length at most $\dim X$.
- A finitely generated module over S has a free resolution of length at most $\dim S$.

CHALLENGE: Unless $X = \mathbb{P}^n$, these resolutions are significantly different.

Products of Projective Space

Fix $r \in \mathbb{N}$, $X := \mathbb{P}^{\vec{n}} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$, and $S := \mathbb{C}[x_{i,j} : 0 \leq i \leq n_j, 1 \leq j \leq r]$, so we have $\dim S - \dim X = r$.

We set $\deg(x_{i,j}) := \vec{e}_j \in \mathbb{Z}^r$ and the irrelevant S-ideal of X is

$$B = \bigcap_{j=1}^{r} \langle x_{0,j}, x_{1,j}, \ldots, x_{n_j,j} \rangle.$$

PROPOSITION: $X = (\operatorname{Spec}(S) \setminus \mathrm{V}(B)) /\!\!/ (\mathbb{C}^*)^r$. Each $\mathbb{Z}^r$-graded S-module gives rise to an $\mathscr{O}_X$-module $\tilde{M}$.

Better Homological Objects

DEFINITION: A $\mathbb{Z}^r$-graded free complex F of S-modules is a **splendid complex** of a $\mathbb{Z}^r$-graded S-module M if the complex $\tilde{F}$ is a resolution of the $\mathscr{O}_X$-module $\tilde{M}$.

PROPOSITION: A $\mathbb{Z}^r$-graded free complex F is splendid if and only if, for all $j \neq 0$, $\mathrm{H}_j(F)$ is a B-torsion module.

PROBLEM: How can we effectively construct short splendid complexes?

Short Splendid Complexes

PROPOSITION: Every $\mathscr{O}_X$-module $\tilde{M}$ has a splendid complex of length at most $\dim X$.

REMARK: Proof uses the resolution of the diagonal $X \hookrightarrow X \times X$, but is not efficient; yields unnecessarily wide complexes. With additional hypothesis on $\tilde{M}$, we get some control over the twists that appear in this splendid complex.

Punctual Example

EXAMPLE: Consider two random points $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$. The minimal free resolution of S/I is

$$S^1 \leftarrow \begin{matrix} S(-2,0)^1 \\ \oplus \\ S(-1,-1)^2 \\ \oplus \\ S(0,-2)^1 \end{matrix} \leftarrow \begin{matrix} S(-2,-1)^2 \\ \oplus \\ S(-1,-2)^2 \end{matrix} \leftarrow S(-2,-2)^1,$$

but there is a splendid complex of the form

$$S^1 \leftarrow S(-1,-1)^2 \leftarrow S(-2,-2)^1.$$

Punctual Subschemes

For $\vec{v} \in \mathbb{N}^r$, set $B^{\vec{v}} := \bigcap_j \langle x_{0,j} \ldots x_{n_j,j} \rangle^{v_j}$.

THEOREM: If $Z \subset X$ with B-saturated ideal I and $\dim Z = 0$, then there is a splendid complex of $\mathcal{O}_Z$ with length $\dim X$ that is the minimal free resolution of $I \cap B^{\vec{v}}$ where $\vec{v} \in \mathbb{N}^r$ and $v_i = 0$ for at least one i.

COROLLARY: If $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$, then $S/(I \cap B^{\vec{v}})$ is Cohen–Macaulay and Hilbert-Burch Theorem describes the corresponding splendid complex.

Multigraded Regularity

DEFINITION: A B-saturated $\mathbb{Z}^r$-graded S-module is $\vec{m}$-regular for some $\vec{m} \in \mathbb{Z}^r$ if $H_B^i(M)_{\vec{m}-\vec{u}} = 0$ for all $i > 0$ and all $\vec{u} \in \mathbb{N}^r$ with $u_1 + u_2 + \cdots + u_r = i - 1$.

REMARK: If $\dim(Z) = 0$, then $\mathscr{O}_Z$ is $\vec{m}$-regular if and only if the Hilbert function and Hilbert polynomial of Z agree at $\vec{m}$.

Winnowing Resolutions

THEOREM: If M is an $\vec{m}$-regular $\mathbb{Z}^r$-graded B-saturated S-module, then the subcomplex of its minimal free resolution consisting of all summand of degree at most $\vec{m}+\vec{n}$ is a splendid complex of $\tilde{M}$.

Winnowing Example

EXAMPLE: Consider 6 random points $Z \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. The minimal free resolution has the form $S^1 \leftarrow S^{37} \leftarrow S^{120} \leftarrow S^{166} \leftarrow S^{120} \leftarrow S^{45} \leftarrow S^7$. The module S/I is $(0,0,2)$-regular and the associated splendid complex has the form

$$S^1 \leftarrow S^{22} \leftarrow S^{51} \leftarrow S^{42} \leftarrow S^{12}.$$