Logarithmic differential forms and questions of residues.

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Differential and combinatorial aspects of singularities.
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Joint work with Mathias Schulze and part of Phd work of Delphine Pol.

References

- Michel Granger and Mathias Schulze, Normal crossing properties of complex hypersurfaces via logarithmic residues. Compositio Math. 150 (2014) 1607-1622.
- Delphine Pol, Logarithmic residues along plane curves. C. R. Math. Acad. Sci. Paris, 353(4):345349, 2015, and Module des résidus logarithmiques des courbes planes, arXiv:1410.2126 (english version in in preparation).
- Delphine Pol, Logarithmic differential forms along complete intersection, work in progress.

Contents

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Definition (K. Saito)

$$\Omega^p(\log D) := \{\omega \in \Omega^p_S(D) \mid d\omega \in \Omega^{p+1}_S(D)\}$$

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All these modules are coherent and reflexive. In particular

$$\Omega^1(\log D) = \mathrm{Hom}_{\mathscr{O}_S}(\mathrm{Der}(-\log D), \mathscr{O}_S), \text{ and}$$
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Definition

The divisor D is free iff $\mathrm{Der}(-\log D)$ or alternatively $\Omega^1(\log D)$ is a free module.

Here are two characterisations of freeness.

Theorem (Saito criterion)

The divisor D is free iff there are $\delta_1, \cdots, \delta_n \in \mathrm{Der}(-\log D)$ such that

$$\det(\delta_1, \cdots, \delta_n) = uh$$

with u a unit. The n-uples $\delta_1, \cdots, \delta_n$ with this property are the generating families of $\mathrm{Der}(-\log D)$.

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Theorem (Terao in qh case, Aleksandrov.)

The following three conditions are equivalent :

1. *The divisor D is free.*
2. *$\mathscr{J}_D$ is Cohen Macaulay (of dimension $n-1$.)*
3. *$\mathscr{O}_\Sigma$ is Cohen Macaulay (of dimension $n-2$.)*

The proof essentially uses the Auslander-Buchsbaum formula.

II. Logarithmic residues

Saito proves that $\omega \in \Omega^p(\log D)$ iff there is $g \in \mathscr{O}_S$ non zero divisor in $\mathscr{O}_D$ and there are holomorphic forms ξ, η such that :

$$g\omega = \frac{dh}{h} \wedge \xi + \eta,$$

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The residue of ω is the meromorphic $(q-1)$-form on D or equivalently on the normalization $\tilde{D}$:

$$\rho_D^p(\omega) := \frac{\xi}{g}|_D \in \Omega_D^{p-1} \otimes Q(\mathcal{O}_D) = \Omega_{\tilde{D}}^{p-1} \otimes Q(\mathcal{O}_{\tilde{D}})$$

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We set $\rho_D^1 = \rho_D$, $\mathscr{R}_D := \rho_D(\Omega^1(\log D)) \subset Q(\mathscr{O}_D)$

Properties of $\mathscr{R}_D$.

Proposition

We have $\mathscr{O}_{\tilde{D}} \subset \mathscr{R}_D$ and there is an exact sequence :

$$0 \longrightarrow \Omega^1_S \longrightarrow \Omega^1(\log D) \xrightarrow{\rho_D} \mathscr{R}_D \longrightarrow 0. \tag{2.1}$$

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By dualizing over $\mathscr{O}_S$ we obtain the following result :

Proposition (G, M. Schulze)

1) There is an exact sequence

$$0 \longrightarrow \mathrm{Der}(-\log D) \longrightarrow \Theta_S \xrightarrow{\sigma_D} \mathscr{R}_D^\vee \longrightarrow \mathrm{Ext}^1_{\mathscr{O}_S}(\Omega^1(\log D), \mathscr{O}_S)$$

2) The image of σ_D is $\mathscr{J}_D \subset \mathscr{R}_D^\vee$ and we always have $\mathscr{R}_D = \mathscr{J}_D^\vee$.
3) When D is free $\mathscr{J}_D = \mathscr{R}_D^\vee$

Ideas for the proof. The presence of $\mathscr{R}_D^\vee$ comes from the *change of ring formula* :

$$\mathscr{R}_D^\vee := \mathrm{Hom}_{\mathscr{O}_D}(\mathscr{R}_D, \mathscr{O}_D) = \mathrm{Hom}_{\mathscr{O}_D}(\mathscr{R}_D, Ext^1_{\mathscr{O}_S}(\mathscr{O}_D, \mathscr{O}_S))$$
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The equality $\sigma_D(\delta)(\rho) = \langle \delta, h \rangle \cdot \rho$ is obtained by studying a diagram built on the complex

$$\operatorname{Hom}_{\mathscr{O}_S}(\Omega^1_S \hookrightarrow \Omega^1(\log D), h\colon \mathscr{O}_S \to \mathscr{O}_S).$$

The $\mathscr{O}_D$ submodules $\mathscr{J}_D$, $\mathscr{R}_D$, $\mathscr{O}_{\tilde{D}}$ of $Q(\mathscr{O}_D)$ are *fractional ideals*, i.e. contain a non zero divisor.
By a result of **De Jong and Van Straten**, duality $I \to I^\vee$ preserves fractional ideals and is an involution on maximal CM ones.
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We may summarize the situation as follows

- We obtain a chain of fractional ideals

$$\mathscr{J}_D \subseteq \mathscr{R}_D^\vee \subseteq \mathscr{C}_D \subseteq \mathscr{O}_D \subseteq \mathscr{O}_{\tilde{D}} \subseteq \mathscr{R}_D$$

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- We obtain a chain of fractional ideals
$$\mathscr{J}_D \subseteq \mathscr{R}_D^\vee \subseteq \mathscr{C}_D \subseteq \mathscr{O}_D \subseteq \mathscr{O}_{\tilde{D}} \subseteq \mathscr{R}_D$$
- If D is free, then $\mathscr{J}_D = \mathscr{R}_D^\vee$ as fractional ideals. In that case :
$$\mathscr{R}_D = \mathscr{O}_{\tilde{D}} \Longleftrightarrow \mathscr{J}_D = \mathscr{C}_D.$$

We call the condition $\mathscr{R}_D = \mathscr{O}_{\tilde{D}}$ the *normal crossing condition*.
The starting point is a result of K. Saito :

Normal crossing condition.

Theorem (Saito)

For a divisor D in a complex manifold S, consider the following conditions:

(A) *the local fundamental groups of the complement $S\setminus D$ are Abelian;*

(B) *in codimension one, that is, outside of an analytic subset of codimension at least 2 in D, D is a normal crossing;*

(C) *the residue of any logarithmic 1-form along D is a weakly holomorphic function on D.*

Then the implications (A) $\Rightarrow$ (B) $\Rightarrow$ (C) *hold true.*

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Theorem (G, Mathias Schulze)

The implication (B) $\Leftarrow$ (C) *in Theorem 3.1 holds true: if the residue of any logarithmic* 1*-form along* D *is a weakly holomorphic function on* D *then* D *is a normal crossing in codimension one.*

Outline of the proof :

- Let $\varphi : Y \to X$ be a morphism. Then

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- Let $\mathscr{R}_\pi := F^0(\Omega_{\tilde{D}/D})$ be the ramification ideal.
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- For a free D it follows :
$\mathscr{R}_D = \mathscr{O}_{\tilde{D}} \Longrightarrow \mathscr{C}_D = \mathscr{J}_D \Longrightarrow \mathscr{R}_\pi = \mathscr{O}_{\tilde{D}} \Longrightarrow \Omega_{\tilde{D}/D} = 0$
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Then $\tilde{D}$ and D have smooth components $\tilde{D}_i \xrightarrow{\simeq} D_i$.
- Finally in codimension one the N.C. condition becomes $\mathscr{R}_D = \oplus_i \mathscr{O}_{D_i}$, a case where the result is known by Saito.

Characterization of normal crossing divisors

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Theorem (E. Faber, G and M. Schulze.)

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- *The ideal* $\mathscr{J}_h = (h'_{x_1}, \cdots, h'_{x_n}) \subset \mathscr{O}_S$ *is a radical ideal*
- *Any of the equivalent conditions* (A), (B), (C),
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Question (E. Faber) In i) or iii), can one get rid of the smoothness hypothesis?

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  - A notion of freeness (G, M. Schulze), and a cohomological characterization of freeness, for multivector fields (G, Schulze) and forms (D. Pol).
  - Analogue of the normal crossing condition has been studied by M. Schulze.

Semigroup of a curve.

Let $(D,0) = \bigcup_{i=1}^{p} D_i \subset (S,0)$ be a reduced curve, with normalization :

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The value of $g \in Q(\mathscr{O}_D)$, is the p-uple of valuations w.r. to t_j's

$$\mathrm{val(g) = (val_1(g), \cdots, val_p(g)) \in (\mathbb{Z} \cup \{\infty\})^p}.$$

Let $\mathrm{val(I) \subset \mathbb{Z}^p}$ be the set of values on non zero divisors in I.

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The semigroup of D is $\Gamma = \mathrm{val}(\mathscr{O}_D) \subset \mathbb{N}^p$.

There is $\underline{\gamma} \in \mathbb{N}^p$ with $\mathrm{val}(\mathscr{C}_D) = (\gamma_1, \cdots, \gamma_p) + \mathbb{N}^p$.

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There is $\underline{\gamma} \in \mathbb{N}^p$ with $\mathrm{val}(\mathscr{C}_{\mathrm{D}}) = (\gamma_1, \cdots, \gamma_{\mathrm{p}}) + \mathbb{N}^{\mathrm{p}}$.
More generally each fractional ideal $I \subset Q(\mathcal{O}_D)$ has a conductor

$$\nu \in \mathbb{Z}^p,\ \mathrm{val(I)} \supset \nu + \mathbb{N}^{\mathrm{p}}$$

A symmetry result.

Theorem (Delgado)

The ring $\mathscr{O}_D$ is Gorenstein iff the semi group Γ has the following property:

$$\forall v \in \mathbb{Z}^p, v \in \Gamma \iff \Delta(\gamma - v - (1, \cdots, 1), \mathscr{O}_D) = \emptyset$$

Here we set $\Delta(\alpha, \mathrm{val}(\mathscr{O}_\mathrm{D})) = \mathrm{val}(\mathscr{O}_\mathrm{D}) \cap \alpha + \Delta_{\mathrm{n}-1}(\mathbb{R}^\mathrm{n}_+)$, $\Delta_{n-1}(\mathbb{R}^n_+)$ is the $n-1$-dimensional part of the upper quadrant $\mathbb{R}^n_+ \supset \mathbb{N}^n_+$

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$\Delta_{n-1}(\mathbb{R}^n_+)$ is the $n-1$-dimensional part of the upper quadrant $\mathbb{R}^n_+ \supset \mathbb{N}^n_+$
Let us restrict to plane curves $D \subset (\mathbb{C}^2, 0)$. We would like to extend this result to a relation between $\mathrm{val}(\mathscr{J}_\mathrm{D})$ and $\mathrm{val}(\mathscr{R}_\mathrm{D})$.

Theorem (D. Pol)

Let D be a plane curve. Then the set of values of the module of logarithmic residues is determined by $\mathscr{J}_D$:

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Remark: The same result holds in a more general context for any Gorenstein reduced curve, and fractionnal ideals $I, I^\vee$.
In particular for a complete intersection curve C, we still have $\mathscr{R}_C, \mathscr{J}_C$ mutually dual. Here $\mathscr{R}_C$ is as defined by Aleksandrov and Tsikh.

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The converse (chasing all possible $v \in \mathscr{V} \setminus \mathrm{val}(\mathscr{R}_{\mathrm{D}})$) is much harder.
D. Pol uses a combinatorial calculation of the dimensions of $\mathscr{R}_D/\mathscr{O}_{\tilde{D}}$ or of $\mathscr{J}_D/C'_D$ quotient of $\mathscr{J}_D$ by its conductor. The relationship with the set $\mathscr{V} \subset \mathbb{Z}^p$ is then the central point.

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D. Pol uses a combinatorial calculation of the dimensions of $\mathscr{R}_D/\mathscr{O}_{\tilde{D}}$ or of $\mathscr{J}_D/C'_D$ quotient of $\mathscr{J}_D$ by its conductor. The relationship with the set $\mathscr{V} \subset \mathbb{Z}^p$ is then the central point.
We give a picture for the case of $f = (x^2 - y^3)(x^4 - y^3)$, $\mu = 19$, $\delta = 10$, $\tau = 17$, $\gamma = (8, 12)$, $\gamma_{\mathscr{J}} = (12, 20)$.

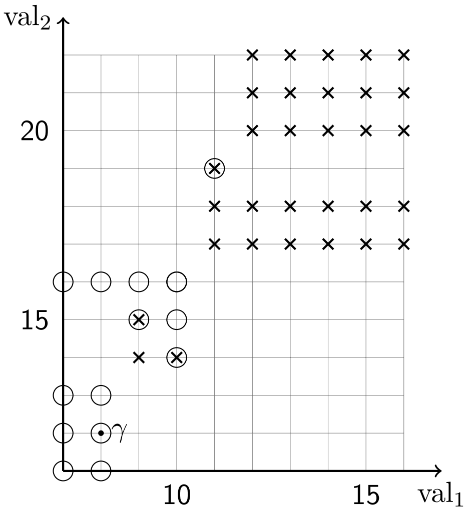

Figure : Semigroup of $\mathscr{I}_D$

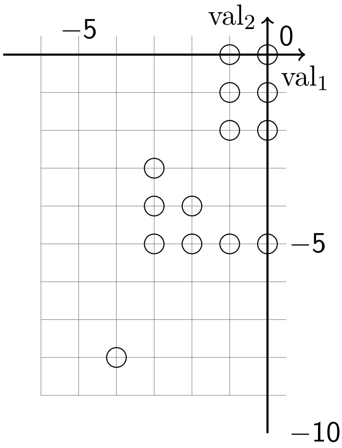

Figure : Multi-values of $\mathscr{R}_D$

An example.

Notice that $\dim \mathscr{R}_D/\mathscr{O}_D = \dim \mathscr{O}_D/\mathscr{J}_D = \tau$ the Tjurina number and $\dim \mathscr{R}_D/\mathscr{O}_{\tilde{D}} = \tau - \delta$.

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Contrary to Γ, $\mathrm{val}(\mathscr{R}_\mathrm{D})$ varies in the μ-constant stratum $S \subset \mathbb{C}^\mu$ of a semiuniversal unfolding F :

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Consider for example the irreducible branch $x^5 - y^6$.

$$F(x, y, s_1, s_2, s_3) = x^5 - y^6 + s_1 x^2 y^4 + s_2 x^3 y^3 + s_3 x^3 y^4,$$

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In the second colum is the value of $\dim_{\mathscr{C}} \mathscr{R}_{D_s} / \mathscr{O}_{\tilde{D}_s} = \tau - 10$

Strate	$\tau - \delta$	values < 0
S_1	10	$-1, -2, -3, -4,\ -7, -8, -9,\ -13, -14,\ -19$
S_2	9	$-1, -2, -3, -4,\ -7, -8, -9,\ -13, -14$
S_3'	8	$-1, -2, -3, -4,\ -7, -8, -9,\ -14$
S_3''	8	$-1, -2, -3, -4,\ -7, -8, -9,\ -13$

Link with Kähler differentials

We consider the module Ω^1_C, and its set of values along a complete intersection curve.

If $\omega = \sum_k a_k dx_k \in \Omega^1_C$, and $t_i \to \varphi_i(t_i)$ parametrizes the ith branch,

$$\mathrm{val_i}(\omega) = 1 + \mathrm{val}\left(\frac{\varphi_i^*(\omega)}{\mathrm{dt_i}}\right) = 1 + \mathrm{val}\left(\sum_k (\mathrm{a_k} \circ \varphi_\mathrm{i})(\mathrm{t_i})\mathrm{x}'_\mathrm{k}(\mathrm{t_i})\right)$$

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Proposition (G, D.Pol)

We have: $\mathrm{val}(\mathscr{J}_C) = \gamma + \mathrm{val}(\Omega^1_C) - (1, \cdots, 1)$

A direct calculation yields $\mathrm{val}(\mathscr{J}_C) = \mathrm{val}(\Omega^1_C) + \underline{\lambda}$ and $\underline{\lambda} = \gamma + (1, \cdots, 1)$ by R. Piene's formula.

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Corollary

For a plane curve $v \in \mathrm{val}(\mathscr{R}_{\mathrm{D}}) \Longleftrightarrow \Delta(-\mathrm{v}, \mathrm{val}(\Omega^1_{\mathrm{D}})) = \emptyset$

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Theorem (Hefez,Hernandez)

The set of analytic classes $\mathscr{M}$ *of plane branches with given topological type satisfies*

$$\mathscr{M} = \bigcup_{\Omega^1_D=\Omega} \mathscr{M}_\Omega.$$

Each $\mathscr{M}_\Omega$ *is separated and a quotient by a finite group of an affine open space.*

A look at complete intersection case.

Let $C \subset S = (\mathbb{C}^n, 0)$ be a complete intersection $f_1 = \cdots = f_k = 0$, D be the hypersurface $f_1 \dots f_k = 0$, and $\widetilde{\Omega_S^q} = \sum_j \frac{1}{f_1 \cdots \widehat{f_j} \cdots f_k} \Omega_S^q$.

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Definition

(Aleksandrov, Tsikh) A form $\omega \in \Omega_S^q(D) := \frac{1}{f_1 \cdots f_k} \Omega_S^q$ is logarithmic iff for all j, $df_j \wedge \omega \in \widetilde{\Omega_S^{q-1}}$. We write $\Omega^q(\log C)$.

Proposition

(Aleksandrov, Tsikh) A form ω is logarithmic iff there are ξ holomorphic, and $\eta \in \widetilde{\Omega_S^{q-1}}$ and $g \in \mathrm{NZD}(\mathscr{O}_\mathrm{C})$ with

$$g\omega = \frac{df_1 \wedge \cdots \wedge df_k}{f_1 \cdots f_k} \wedge \xi + \eta$$

We define the residue of ω as $\operatorname{res} \omega = \frac{\xi}{g}|_C \in \Omega^{q-k} \otimes Q(\mathscr{O}_C)$.

Definition (G, M. Schulze)

The complete intersection is called free iff $\mathscr{I}_C$ (or $\mathscr{O}_C/\mathscr{I}_C$) is maximal Cohen Macaulay.

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We set $\mathscr{R}_C = res(\Omega^k(\log C)) \subset Q(\mathscr{O}_C)$. Its does not depend on the choice of f_i's. By direct calculation or because $\mathscr{R}_C =$ the set of regular 0-forms (Aleksandrov).

We end with a few facts about $\mathscr{R}_C$, noticed or proved in G, M.Schulze, or in D. Pol;

- We have $\mathscr{J}_C^\vee = \mathscr{R}_C \supset \mathscr{O}_{\tilde{C}}$, and for a free C also $\mathscr{R}_C^\vee = \mathscr{J}_C$.

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- For a complete intersection curve this yields a result of symmetry, completely similar to the planar case.
- We can see that $\Omega^q(\log D) \subset \Omega^q(\log C)$. In particular

$$res_C(\Omega^k(\log D)) \subset \mathscr{R}_C$$

but in general this inclusion is strict (Example of D. Pol).

Thank you for your attention