Logarithmic differential forms and questions of residues.

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Differential and combinatorial aspects of singularities.
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Joint work with Mathias Schulze and part of Phd work of Delphine Pol.
References

- Delphine Pol, Logarithmic differential forms along complete intersection, work in progress.
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I. Basic definitions.

Let $D \subset S = (\mathbb{C}^n, 0)$ be a reduced effective divisor, $\mathcal{I}_D = \mathcal{O}_S \cdot h$ its defining ideal. We set:

$$\Theta_S := \text{Der}(\mathcal{O}_S) = \text{Hom}_{\mathcal{O}_S}(\Omega^1_S, \mathcal{O}_S),$$

$$\Omega^p_S(\log D) := \{ \omega \in \Omega^p_S(D) \mid d\omega \in \Omega^{p+1}_S(D) \},$$

$$\text{Der}(-\log D) := \{ \delta \in \Theta_S \mid dh(\delta) \in \mathcal{I}_D \}.$$
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**Definition (K. Saito)**

$$\Omega^p(\log D) := \{ \omega \in \Omega^p_S(D) \mid d\omega \in \Omega^{p+1}_S(D) \}$$

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All these modules are coherent and reflexive. In particular

$$\Omega^1(\log D) = \text{Hom}_{\mathcal{O}_S}(\text{Der}(-\log D), \mathcal{O}_S), \quad \text{and}$$

$$\text{Der}(-\log D) = \text{Hom}_{\mathcal{O}_S}(\Omega^1(\log D), \mathcal{O}_S)$$
Let $\Sigma = \text{Sing}(D)$, with $\mathcal{O}_\Sigma = \mathcal{O}_S/(J(h), h) = \mathcal{O}_D/\mathcal{J}_D$. We have the following exact sequences:
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\[ (1.1) \quad 0 \rightarrow \text{Der}(-\log D) \rightarrow \Theta_S \rightarrow^{dh} \mathcal{I}_D \rightarrow 0 \]
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(1.1) \[ 0 \rightarrow \text{Der}(-\log D) \rightarrow \Theta_S \xrightarrow{dh} \mathcal{I}_D \rightarrow 0 \]

(1.2) \[ 0 \rightarrow \text{Der}(-\log D) \rightarrow \Theta_S \oplus \mathcal{O}_S \xrightarrow{dh,-h} \mathcal{O}_S \rightarrow \mathcal{O}_\Sigma \rightarrow 0 \]

In sequence (1.2) the first arrow is: $\delta \mapsto (\delta, \delta(h)/h)$
Let $\Sigma = \text{Sing}(D)$, with $\mathcal{O}_\Sigma = \mathcal{O}_S/(J(h), h) = \mathcal{O}_D/\mathcal{I}_D$. We have the following exact sequences:

(1.1) \hspace{1cm} 0 \longrightarrow \text{Der}(−\log D) \longrightarrow \Theta_S \overset{dh}{\longrightarrow} \mathcal{I}_D \longrightarrow 0

(1.2) \hspace{1cm} 0 \longrightarrow \text{Der}(−\log D) \longrightarrow \Theta_S \oplus \mathcal{O}_S \overset{dh,−h}{\longrightarrow} \Theta_S \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0

In sequence (1.2) the first arrow is: $\delta \mapsto (\delta, \delta(h)/h)$

**Definition**

The divisor $D$ is free iff $\text{Der}(−\log D)$ or alternatively $\Omega^1(\log D)$ is a free module.

Here are two characterisations of freeness.
**Theorem (Saito criterion)**

*The divisor $D$ is free iff there are $\delta_1, \cdots, \delta_n \in \text{Der}(-\log D)$ such that*

\[
\det(\delta_1, \cdots, \delta_n) = uh
\]

*with $u$ a unit. The $n$-uples $\delta_1, \cdots, \delta_n$ with this property are the generating families of $\text{Der}(-\log D)$.***
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**Theorem (Terao in qh case, Aleksandrov.)**

The following three conditions are equivalent:

1. The divisor $D$ is free.
2. $\mathcal{I}_D$ is Cohen Macaulay (of dimension $n - 1$).
3. $\mathcal{O}_\Sigma$ is Cohen Macaulay (of dimension $n - 2$).

The proof essentially uses the Auslander-Buchsbaum formula.
II. Logarithmic residues

Saito proves that $\omega \in \Omega^p(\log D)$ iff there is $g \in \mathcal{O}_S$ non zero divisor in $\mathcal{O}_D$ and there are holomorphic forms $\xi, \eta$ such that:

$$g\omega = \frac{dh}{h} \wedge \xi + \eta,$$
Saito proves that \( \omega \in \Omega^p(\log D) \) iff there is \( g \in \mathcal{O}_S \) non zero divisor in \( \mathcal{O}_D \) and there are holomorphic forms \( \xi, \eta \) such that:

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g\omega = \frac{dh}{h} \wedge \xi + \eta,
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**Definition**

The residue of \( \omega \) is the meromorphic \((q - 1)\)-form on \( D \) or equivalently on the normalization \( \tilde{D} \):

\[
\rho^p_D(\omega) := \frac{\xi}{g}|_D \in \Omega^{p-1}_D \otimes Q(\mathcal{O}_D) = \Omega^{p-1}_\tilde{D} \otimes Q(\mathcal{O}_{\tilde{D}})
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**Definition**

The residue of $\omega$ is the meromorphic $(q-1)$-form on $D$ or equivalently on the normalization $\tilde{D}$:

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We set $\rho^1_D = \rho_D$, $\mathcal{R}_D := \rho_D(\Omega^1(\log D)) \subset Q(\mathcal{O}_D)$.
Properties of $\mathcal{R}_D$.

Proposition

We have $\mathcal{O}_D \subset \mathcal{R}_D$ and there is an exact sequence:

$$0 \to \Omega^1_S \to \Omega^1(\log D) \xrightarrow{\rho_D} \mathcal{R}_D \to 0.$$
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Proposition

We have $\mathcal{O}_D \subset \mathcal{R}_D$ and there is an exact sequence:

\[
0 \rightarrow \Omega_S^1 \rightarrow \Omega^1(\log D) \xrightarrow{\rho_D} \mathcal{R}_D \rightarrow 0.
\]

By dualizing over $\mathcal{O}_S$ we obtain the following result:

Proposition (G, M. Schulze)

1) There is an exact sequence

\[
0 \rightarrow \text{Der}(-\log D) \rightarrow \Theta_S \xrightarrow{\sigma_D} \mathcal{R}_D^\vee \rightarrow \text{Ext}^1_{\mathcal{O}_S}(\Omega^1(\log D), \mathcal{O}_S)
\]

2) The image of $\sigma_D$ is $\mathcal{J}_D \subset \mathcal{R}_D^\vee$ and we always have $\mathcal{R}_D = \mathcal{J}_D^\vee$.

3) When $D$ is free $\mathcal{J}_D = \mathcal{R}_D^\vee$.
Ideas for the proof. The presence of $R_D^\vee$ comes from the change of ring formula:

$$R_D^\vee := \text{Hom}_{O_D}(R_D, O_D) = \text{Hom}_{O_D}(R_D, \text{Ext}^1_{O_S}(O_D, O_S))$$

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Ideas for the proof. The presence of $\mathcal{R}^\vee_D$ comes from the *change of ring formula*:

$$\mathcal{R}^\vee_D := \text{Hom}_{\mathcal{O}_D}(\mathcal{R}_D, \mathcal{O}_D) = \text{Hom}_{\mathcal{O}_D}(\mathcal{R}_D, \text{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_D, \mathcal{O}_S))$$

$$= \text{Ext}^1_{\mathcal{O}_S}(\mathcal{R}_D, \mathcal{O}_S)$$

The equality $\sigma_D(\delta)(\rho) = \langle \delta, h \rangle \cdot \rho$ is obtained by studying a diagram built on the complex

$$\text{Hom}_{\mathcal{O}_S}(\Omega^1_S \hookrightarrow \Omega^1(\log D), h: \mathcal{O}_S \rightarrow \mathcal{O}_S).$$
The $\mathcal{O}_D$ submodules $\mathcal{I}_D$, $\mathcal{R}_D$, $\mathcal{O}_D$ of $Q(\mathcal{O}_D)$ are fractional ideals, i.e. contain a non zero divisor.

By a result of De Jong and Van Straten, duality $I \rightarrow I^\vee$ preserves fractional ideals and is an involution on maximal CM ones.

This is the case for the conductor $\mathcal{C}_D := \mathcal{O}_D^\vee$.
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We may summarize the situation as follows

- We obtain a chain of fractional ideals

$$\mathcal{I}_D \subseteq \mathcal{R}_D^\vee \subseteq \mathcal{C}_D \subseteq \mathcal{O}_D \subseteq \mathcal{O}_{\tilde{D}} \subseteq \mathcal{R}_D$$
The $O_D$ submodules $I_D$, $R_D$, $O_D$ of $Q(O_D)$ are fractional ideals, i.e. contain a non zero divisor.

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We may summarize the situation as follows

- We obtain a chain of fractional ideals
  \[ I_D \subseteq R_D^\vee \subseteq C_D \subseteq O_D \subseteq O_D^\vee \subseteq R_D \]

- If $D$ is free, then $I_D = R_D^\vee$ as fractional ideals. In that case:
  \[ R_D = O_D^\vee \iff I_D = C_D. \]

We call the condition $R_D = O_D^\vee$ the normal crossing condition.

The starting point is a result of K. Saito:
Theorem (Saito)

For a divisor $D$ in a complex manifold $S$, consider the following conditions:

(A) the local fundamental groups of the complement $S \setminus D$ are Abelian;

(B) in codimension one, that is, outside of an analytic subset of codimension at least 2 in $D$, $D$ is a normal crossing;

(C) the residue of any logarithmic 1-form along $D$ is a weakly holomorphic function on $D$.

Then the implications $(A) \Rightarrow (B) \Rightarrow (C)$ hold true.
In his 1980 paper Saito asked for the converse implications:
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The implication $(A) \iff (B)$ in Theorem 3.1 holds true.

**Theorem (G, Mathias Schulze)**

The implication $(B) \iff (C)$ in Theorem 3.1 holds true: if the residue of any logarithmic 1-form along $D$ is a weakly holomorphic function on $D$ then $D$ is a normal crossing in codimension one.
Outline of the proof:

- Let $\varphi : Y \to X$ be a morphism. Then

  $$\Omega_{Y/X} = 0 \iff \varphi \text{ is an immersion}.$$
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- In codimension 1, $D$ is free, $\tilde{D}$ is smooth.
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- In codimension 1, $D$ is free, $\tilde{D}$ is smooth.

- Let $\mathcal{R}_\pi := F^0(\Omega_{\tilde{D}/D})$ be the ramification ideal.
  By a formula of Ragni Piene:
  \[ \mathcal{C}_D \mathcal{R}_\pi = \mathcal{I}_D \mathcal{O}_{\tilde{D}} \]
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- Let \( \mathcal{R}_\pi := F^0(\Omega_{\tilde{D}/D}) \) be the ramification ideal.

  By a formula of Ragni Piene:
  \[ C_D \mathcal{R}_\pi = \mathcal{I}_D \mathcal{O}_{\tilde{D}} \]

- For a free \( D \) it follows:
  \[ \mathcal{R}_D = \mathcal{O}_{\tilde{D}} \implies C_D = \mathcal{I}_D \implies \mathcal{R}_\pi = \mathcal{O}_{\tilde{D}} \implies \Omega_{\tilde{D}/D} = 0 \]

Then \( \tilde{D} \) and \( D \) have smooth components \( \tilde{D}_i \xrightarrow{\sim} D_i \).
Outline of the proof:

- Let $\varphi : Y \to X$ be a morphism. Then
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  R_D = \mathcal{O}_{\tilde{D}} \implies C_D = \mathcal{J}_D \implies R_{\pi} = \mathcal{O}_{\tilde{D}} \implies \Omega_{\tilde{D}/D} = 0
  \]
  Then $\tilde{D}$ and $D$ have smooth components $\tilde{D}_i \xrightarrow{\sim} D_i$.

- Finally in codimension one the N.C. condition becomes $R_D = \bigoplus_i \mathcal{O}_{D_i}$, a case where the result is known by Saito.
Characterization of normal crossing divisors

A normal crossing divisor is free and $\mathcal{I}_D$ is a radical ideal of $\mathcal{O}_D$. A partial converse is:
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**Theorem (E. Faber, G and M. Schulze.)**

For a free divisor with smooth normalization, any of the conditions

- The ideal $\mathcal{I}_h = (h'_{x_1}, \cdots, h'_{x_n}) \subset \mathcal{O}_S$ is a radical ideal
- Any of the equivalent conditions (A), (B), (C),
- The Jacobian ideal $\mathcal{I}_D$ is radical.

imply that $D$ is a normal crossing divisor.
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Question (E. Faber) In i) or iii), can one get rid of the smoothness hypothesis?

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- When $D$ does not satisfy (A), (B) or (C), How to determine $R_D \not\subset \mathcal{O}_D$
- Case of curves: Detailed answer in terms of the semigroup of multivaluations (or values). Delphine Pol.
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- Description of the dual residue module $\mathcal{R}^\vee_C$. Results similar to the above for curves, and other examples (D. Pol).
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  - Description of the dual residue module $\mathcal{R}_C^\vee$. Results similar to the above for curves, and other examples (D. Pol).
  - A notion of freeness (G, M. Schulze), and a cohomological characterization of freeness, for multivector fields (G, Schulze) and forms (D. Pol).
Going further.

- When $D$ does not satisfy (A), (B) or (C), How to determine $\mathcal{R}_D \supsetneq \mathcal{O}_\tilde{D}$
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- There is a notion of multiresidues along a complete intersection due to Aleksandrov and Tsikh.
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  - A notion of freeness (G, M. Schulze), and a cohomological characterization of freeness, for multivector fields (G, Schulze) and forms (D. Pol).
  - Analogue of the normal crossing condition has been studied by M. Schulze.
Semigroup of a curve.

Let \((D, 0) = \bigcup_{i=1}^{p} D_i \subset (S, 0)\) be a reduced curve, with normalization:

\[
\mathcal{O}_D \rightarrow \mathcal{O}_{\tilde{D}} \simeq \mathbb{C}\{t_1\} \oplus \cdots \oplus \mathbb{C}\{t_p\}.
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The value of \(g \in \mathbb{Q}(\mathcal{O}_D)\), is the \(p\)-uple of valuations w.r. to \(t_j\)'s

\[
\text{val}(g) = (\text{val}_1(g), \cdots, \text{val}_p(g)) \in (\mathbb{Z} \cup \{\infty\})^p.
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Let \(\text{val}(I) \subset \mathbb{Z}^p\) be the set of values on non zero divisors in \(I\).
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**Definition**

The semigroup of \(D\) is \(\Gamma = \text{val}(\mathcal{O}_D) \subset \mathbb{N}^p\).

There is \(\gamma \in \mathbb{N}^p\) with \(\text{val}(\mathcal{C}_D) = (\gamma_1, \cdots, \gamma_p) + \mathbb{N}^p\).
Let \((D, 0) = \bigcup_{i=1}^{p} D_i \subset (S, 0)\) be a reduced curve, with normalization:

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More generally each fractional ideal \(I \subset Q(\mathcal{O}_D)\) has a conductor

\[ \nu \in \mathbb{Z}^p, \text{val}(I) \supset \nu + \mathbb{N}^p \]
Theorem (Delgado)

The ring $\mathcal{O}_D$ is Gorenstein iff the semi group $\Gamma$ has the following property:

$$\forall v \in \mathbb{Z}^p, v \in \Gamma \iff \Delta(\gamma - v - (1, \cdots, 1), \mathcal{O}_D) = \emptyset$$

Here we set $\Delta(\alpha, \text{val}(\mathcal{O}_D)) = \text{val}(\mathcal{O}_D) \cap \alpha + \Delta_{n-1}(\mathbb{R}_+^n)$, $\Delta_{n-1}(\mathbb{R}_+^n)$ is the $n-1$-dimensional part of the upper quadrant $\mathbb{R}_+^n \supset \mathbb{N}_+^n$. 

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Let us restrict to plane curves $D \subset (\mathbb{C}^2, 0)$. We would like to extend this result to a relation between $\text{val}(\mathcal{I}_D)$ and $\text{val}(\mathcal{R}_D)$. 

Theorem (D. Pol)

Let $D$ be a plane curve. Then the set of values of the module of logarithmic residues is determined by $\mathcal{I}_D$: 

$$v \in \text{val}(\mathcal{R}_D) \iff \Delta(\gamma - v - (1, \cdots, 1), \mathcal{I}_D) = \emptyset$$
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Remark: The same result holds in a more general context for any Gorenstein reduced curve, and fractional ideals $I, I^\vee$. In particular for a complete intersection curve $C$, we still have $\mathcal{R}_C, \mathcal{I}_C$ mutually dual. Here $\mathcal{R}_C$ is as defined by Aleksandrov and Tsikh.
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$$\text{val}(\mathcal{H}_D) \subset \mathcal{V} := \{ v \mid \Delta(\gamma - v - (1, \cdots, 1), \mathcal{J}_D) = \emptyset \}$$
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The converse (chasing all possible \( v \in \mathcal{V} \setminus \text{val}(\mathcal{R}_D) \)) is much harder.

D. Pol uses a combinatorial calculation of the dimensions of \( \mathcal{R}_D / \mathcal{O}_\tilde{D} \) or of \( \mathcal{I}_D / \mathcal{C}'_D \) quotient of \( \mathcal{I}_D \) by its conductor. The relationship with the set \( \mathcal{V} \subset \mathbb{Z}^p \) is then the central point.
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The converse (chasing all possible \( v \in \mathcal{V} \setminus \text{val}(\mathcal{R}_D) \)) is much harder.

D. Pol uses a combinatorial calculation of the dimensions of \( \mathcal{R}_D / \mathcal{O}_\tilde{D} \) or of \( \mathcal{I}_D / \mathcal{C}_D' \) quotient of \( \mathcal{I}_D \) by its conductor. The relationship with the set \( \mathcal{V} \subset \mathbb{Z}^p \) is then the central point.
We give a picture for the case of \( f = (x^2 - y^3)(x^4 - y^3) \), \( \mu = 19 \), \( \delta = 10 \), \( \tau = 17 \), \( \gamma = (8, 12) \), \( \gamma \mathcal{I} = (12, 20) \).
Figure: Semigroup of $\mathcal{J}(D)$
Basic definitions.
Logarithmic residues and duality.
Normal crossing conditions
Residues along plane curves
The complete intersection case.

**Figure**: Multi-values of $\mathcal{R}_D$
An example.

Notice that $\dim \mathcal{R}_D/\mathcal{O}_D = \dim \mathcal{O}_D/J_D = \tau$ the Tjurina number and $\dim \mathcal{R}_D/\mathcal{O}_D = \tau - \delta$. 
An example.

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Contrary to $\Gamma$, $\text{val}(\mathcal{R}_D)$ varies in the $\mu$-constant stratum $S \subset \mathbb{C}^\mu$ of a semiuniversal unfolding $F : \mathbb{C}^2 \times \mathbb{C}^\mu \to \mathbb{C}$, $F(x, y, s) = f(x, y) + \sum_{1 \leq k \leq \mu} t^k m_k(x, y)$.
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In fact $\tau$-constant deformations are adapted to residue calculations as flat deformations of $f, f'_x, f'_y$: 

\[ \mathcal{O}^2 \times \mathbb{C}^\mu \to \mathbb{C}, F(x, y, s) = f(x, y) + \sum_{1 \leq k \leq \mu} t^k m_k(x, y) \]
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\[ \text{Proposition} \]

The partition of \( S \) by \( \mathrm{val}(\mathcal{R}_D) \) is a partition into analytic locally closed subsets refining the \( \tau \)-constant strata.
An example.

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**Proposition**

*The partition of \( S \) by \( \text{val}(\mathcal{R}_D) \) is a partition into analytic locally closed subsets refining the \( \tau \)-constant strata.*

Consider for example the irreducible branch \( x^5 - y^6. \)
Basic definitions.
Logarithmic residues and duality.
Normal crossing conditions
Residues along plane curves
The complete intersection case.

\[ F(x, y, s_1, s_2, s_3) = x^5 - y^6 + s_1 x^2 y^4 + s_2 x^3 y^3 + s_3 x^3 y^4, \]

\[ \mu = 20, \delta = 10, \]
Basic definitions.  
Logarithmic residues and duality.  
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\[ S_1 = \{0\}, \]

\[ S_2 = \{(0, 0, s_3), s_3 \neq 0\} \]

\[ S'_3 = \{(s_1, s_2, s_3), s_1 \neq 0\} \]

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In the second column is the value of \( \dim_{\mathcal{O}} \mathcal{R}_{D_s}/\mathcal{O}_{\tilde{D}_s} = \tau - 10 \)

<table>
<thead>
<tr>
<th>Strate</th>
<th>( \tau - \delta )</th>
<th>values &lt; 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1 )</td>
<td>10</td>
<td>(-1, -2, -3, -4, -7, -8, -9, -13, -14, -19)</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>9</td>
<td>(-1, -2, -3, -4, -7, -8, -9, -13, -14)</td>
</tr>
<tr>
<td>( S'_3 )</td>
<td>8</td>
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We consider the module $\Omega^1_C$, and its set of values along a complete intersection curve.

If $\omega = \sum_k a_k dx_k \in \Omega^1_C$, and $t_i \rightarrow \varphi_i(t_i)$ parametrizes the $i$th branch,

$$\text{val}_i(\omega) = 1 + \text{val} \left( \frac{\varphi_i^*(\omega)}{dt_i} \right) = 1 + \text{val} \left( \sum_k (a_k \circ \varphi_i)(t_i) x'_k(t_i) \right)$$
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**Proposition (G, D.Pol)**

*We have: $\text{val}(\mathcal{C}) = \gamma + \text{val}(\Omega^1_C) - (1, \cdots, 1)$*

A direct calculation yields $\text{val}(\mathcal{C}) = \text{val}(\Omega^1_C) + \lambda$
and $\lambda = \gamma + (1, \cdots, 1)$ by R. Piene’s formula.
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**Proposition (G, D.Pol)**

We have: 

$$\text{val}(\mathcal{I}_C) = \gamma + \text{val}(\Omega^1_C) - (1, \cdots, 1)$$

A direct calculation yields 

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Corollary

For a plane curve $v \in \text{val}(\mathcal{R}_D) \iff \Delta(-v, \text{val}(\Omega^1_D)) = \emptyset$

In this way $\mathcal{R}_D$ is related to the problem of moduli space for plane branches with a given semi group $\Gamma$.
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Theorem (Hefez, Hernandez)

*The set of analytic classes* $\mathcal{M}$ *of plane branches with given topological type satisfies*

$$\mathcal{M} = \bigcup_{\Omega_D = \Omega} \mathcal{M}_\Omega.$$

*Each* $\mathcal{M}_\Omega$ *is separated and a quotient by a finite group of an affine open space.*
A look at complete intersection case.

Let $C \subset S = (\mathbb{C}^n, 0)$ be a complete intersection $f_1 = \cdots = f_k = 0$, $D$ be the hypersurface $f_1 \cdots f_k = 0$, and $\widetilde{\Omega}^q_S = \sum_j \frac{1}{f_1 \cdots \hat{f}_j \cdots f_k} \Omega^q_S$. 
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Definition

(Aleksandrov, Tsikh) A form $\omega \in \Omega_S^q(D) := \frac{1}{f_1 \cdots f_k} \Omega_S^q$ is logarithmic iff for all $j$, $df_j \wedge \omega \in \Omega_S^{q-1}$. We write $\Omega^q(\log C)$.

Proposition

(Aleksandrov, Tsikh) A form $\omega$ is logarithmic iff there are $\xi$ holomorphic, and $\eta \in \Omega_S^{q-1}$ and $g \in \text{NZD}(\mathcal{O}_C)$ with

$$g \omega = \frac{df_1 \wedge \cdots \wedge df_k}{f_1 \cdots f_k} \wedge \xi + \eta$$

We define the residue of $\omega$ as $\text{res} \omega = \frac{\xi}{g} \big|_C \in \Omega^{q-k} \otimes Q(\mathcal{O}_C)$. 
Definition (G, M. Schulze)

The complete intersection is called free iff $\mathcal{J}_C$ (or $\mathcal{O}_C / \mathcal{J}_C$) is maximal Cohen Macaulay.
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This property is equivalent to: $\text{Der}^k(-\log C) := \ker(\Theta_S^k \rightarrow \mathcal{I}_C)$ is of projective dimension $k - 1$. 
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Theorem (Delphine Pol)

*The complete intersection \( C \) is free iff \( \Omega^k(\log C) \) is of projective dimension \( k - 1 \).*

We set \( \mathcal{R}_C = \text{res}(\Omega^k(\log C)) \subset Q(\mathcal{O}_C) \). Its does not depend on the choice of \( f_i \)'s. By direct calculation or because \( \mathcal{R}_C \) = the set of regular 0-forms (Aleksandrov).
We end with a few facts about $R_C$, noticed or proved in G, M. Schulze, or in D. Pol;

- We have $J_C^\vee = R_C \supset \mathcal{O}_{\tilde{C}}$, and for a free $C$ also $R_C^\vee = J_C$. 
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- For a complete intersection curve this yields a result of symmetry, completely similar to the planar case.
- We can see that $\Omega^q(\log D) \subset \Omega^q(\log C)$. In particular

$$\text{res}_C(\Omega^k(\log D)) \subset \mathcal{R}_C$$

but in general this inclusion is strict (Example of D. Pol).
Thank you for your attention