Annihilating powers of arrangements

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$R = \mathbb{C}[x_1, \ldots, x_n]$	polynomial ring
$f \in R$	polynomial, not constant
$J = (f, f_1, \ldots, f_n)$	ideal of singular locus
$W = R\langle \partial_1, \ldots, \partial_n \rangle$	Weyl algebra
$W[s]$	adjoining a new variable

Definition

Let $W[s]$ act on $R[f^{-1}, s] \cdot f^s$ by the formal chain rule:

$$\partial_i \bullet (gf^s) = (f\partial_i(g) + gs\partial_i(f))f^{s-1}$$

for all $g \in R[f^{-1}, s]$. Denote by $W[s] \bullet f^s$ the $W[s]$-submodule generated by $1 \cdot f^s$.

Some things one might hope for:

Let $\mathrm{ann}_{W[s]}(f^s)$ be the elements in $W[s]$ that kill f^s.
Note: f homogeneous of degree d

$$\Rightarrow \underbrace{x_1\partial_1 + \ldots + x_n\partial_n}_{E} - ds \in \mathrm{ann}_{W[s]}(f^s).$$

1. Is $R[f^{-1}, s]f^s = W[s] \bullet f^s$?
   No: even with $f = x$, $W[s] \bullet f^s \not\ni f^{s-1}$ (but sf^{s-1}).
2. Does $W[s] \bullet f^s$ contain sf^{s-1} (or $s^k f^{s-1}$ for some k)?
   No: $f = xy$ (or $f = xy(x+y)$—a bit tedious by hand).
3. For homogeneous f, is $\mathrm{ann}_{W[s]}(f^s)$ generated by $\{f_i\partial_j - f_j\partial_i\}$ and $E - s$?
   No: for $f = x^2y$, $x\partial_x - 2y\partial_y \in \mathrm{ann}_{W[s]}(f^s)$.
4. Is $\mathrm{ann}(f^s)$ generated by derivations?
   No: $x^5 + x^4y + y^4$ (needs a computer).
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Bernstein–Sato polynomial

What is the difference between $W[s] \bullet f^s$ and $W[s] \bullet f^{s+1}$? Surprisingly,

Theorem ((Sato,) Bernstein (, Björk))

$\exists\, P \in W[s]$, $\exists 0 \neq b_P \in \mathbb{C}[s]$,

$$P(x, \partial, s) \bullet f^{s+1} = b_P(s) f^s.$$

(Means: $W[s] \bullet f^s / W[s] \bullet f^{s+1}$ is killed by $b_P(s)$).

Definition

The (monic) generator of the ideal $\{b_P\}$ is the Bernstein–Sato polynomial $b_f(s)$.

Some examples

- $f = x_1$ (or smooth): $P = \partial_1$, $b_f(s) = s + 1$.
- $f = x_1 x_2 \cdots x_k$: $P = \partial_1 \partial_2 \cdots \partial_k$, $b_f(s) = (s+1)^k$.
- $f = x_1^2 + x_2^2 + x_3^2$: $P = (\partial_1^2 + \partial_2^2 + \partial_3^2)/4$, $b_f(s) = (s+1)(2s+3)/2$.
- $f = \det(x_{i,j})_1^n$: $P = \det(\partial_{i,j})_1^n$, $b_f(s) = (s+1)\cdots(s+n)$. (Cayley)
- $f = x^2 + y^3$: $P =$somewhat long, $b_f(s) = (s+5/6)(s+1)(s+7/6)$.
- $f = (x+y)\cdots(x+ky)$: $P =$rather long, $b_f(s)/(s+1) = (s+2/k)\cdots(s+(2k-2)/k)$.
- $f = x^5 + x^4 y + y^4$: $P =$terrifying, $b_f(s)/(s+1) = (10s+7)(10s+9)(10s+11)(10s+13)(20s+9)(20s+11)(20s+13)(20s+17)(20s+19)(20s+21)(20s+23)(20s+27)$.

Classical results

ρ_f =root set of $b_f(s)$

Theorem

- $b_f(-1) = 0$
- $b_f(s) \in \mathbb{Q}[s]$
  in fact, $\rho_f \subseteq \mathbb{Q}_-$ *(Malgrange/Kashiwara)*
- $\rho_f \subseteq (-n, 0)$ *(Saito)*
- *isol sing:* ρ_f *=e-vals of some operator on* R/J *(Malgrange)*
- *isolated and* w*-homogeneous:* $-\rho_f = \deg_w((R/J)\frac{dx}{f}) \cup \{1\}$

Also relates to: lct, multiplier ideals, periods, p-adic stuff, ...

Conjectures on annihilators

Terao

For any (reduced) arrangement, the D-annihilator of $1/f$ is generated by derivations.

Torrelli; -

For any (reduced) arrangement, the $D[s]$-annihilator of f^s is generated by derivations.

Note: the second implies the first.
(Kashiwara: plugging $-k \ll 0$ into $\mathrm{ann}_{D[s]}(f^s)$ gives $\mathrm{ann}_D(1/f^k)$;
Leykin: $-1 \ll 0$ for arrangements).

An interesting complex

Logarithmic differntial forms

$$\Omega^i(\log f) = \{\omega \in \tfrac{1}{f}\Omega^i \mid \omega \wedge \tfrac{df}{f} \in \tfrac{1}{f}\Omega^{i+1}\}.$$

$$\Omega^i_0(\log f) = \{\omega \in \tfrac{1}{f}\Omega^i \mid \omega \wedge \tfrac{df}{f} = 0\}.$$

Usually: $\Omega^i(\log f)$ complex with de Rham differential.

Cotangent map: $T^*X = \operatorname{Spec} R[y] \twoheadrightarrow \operatorname{Spec} R = X$.

$$(*) \quad \underbrace{\Omega^0_0(\log f)}_{=0}[y] \overset{y\,\mathrm{dx}}{\to} \underbrace{\Omega^1_0(\log f)}_{\cong R\cdot df/f}[y] \overset{y\,\mathrm{dx}}{\to} \ldots \overset{y\,\mathrm{dx}}{\to} \underbrace{\Omega^n_0(\log f)}_{\cong R\cdot 1/f}[y]$$

$H^n \cong R[y]/\mathcal{L}_f$ where

$$\mathcal{L}_f = \{\sum_{i=1}^n a_i y_i \mid \sum_{i=1}^n a_i f_i = 0\}.$$

Exactness of complex $(*)$

Theorem (-)

- *Exact in positions* $0, 1, 2$.
- *Resolution if* $n \leq 4$*; possibly always.*
- *IF* f *is locally strongly Euler homogeneous, and tame, and Saito-holonomic,*
  THEN $(*)$ *is a resolution and* $\mathcal{L}_f$ *is a Cohen–Macaulay prime ideal.*

Zeta functions

We are over $\mathbb{C}$, so have embedded resolution of singularities:

$$\pi\colon (Y \supseteq \tilde{X}) \to (\mathbb{C}^n \supseteq X = \mathrm{Var}(f)).$$

Let $\tilde{X} = \bigcup_{i \in S} E_i$, and for $I \subseteq S$,

$$E_I^* := \bigcap_{i \in I} E_i \setminus \bigcup_{j \notin I} E_j.$$

Definition

The topological zeta function of f is

$$Z_f(s) = \sum_I \chi(E_I^*) \prod \frac{1}{N_i s + \nu_i}$$

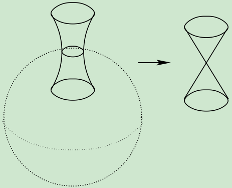
where N_i is the multiplicity of E_i in $\mathrm{Div}(f \circ \pi)$ and $\nu_i - 1$ is the multiplicity of E_i in $\mathrm{Div}(\pi^*(dx_1 \wedge \ldots \wedge dx_n))$.

Denef, Loeser: this is independent of the resolution.

A zeta function example

Example (A zeta function)

- Let $f = x_1^2 + x_2^2 + x_3^2$ in $\mathbb{C}[x_1, x_2, x_3]$.
- Then $S = \{1, 2\}$, E_1 is the strict transform and $E_2 = \mathbb{P}^2$.
- $E_1^* = \mathrm{Var}(f) \setminus \{0\}$,
  $E_{1,2}^* = \mathrm{Proj}(f) \cong \mathbb{P}^1$,
  $E_2^* = \mathbb{P}^2 \smallsetminus \mathrm{Proj}(f)$



- $\chi(E_1^*) = 0$, $\chi(E_{1,2}^*) = 2$, $\chi(E_2^*) = 3 - 2 = 1$
- $N_1 = 1$, $N_2 = 2$, $\nu_1 = 1$, $\nu_2 = 3$.
- $Z_f(s) = 0 + \dfrac{1}{2 \cdot s + 3} + 2\dfrac{1}{1 \cdot s + 1} \cdot \dfrac{1}{2 \cdot s + 3} = \dfrac{s+3}{(s+1)(2s+3)}$.

The monodromy conjecture

Conjecture

(SMC) *If α is a pole of $Z_f(s)$ then $b_f(\alpha) = 0$.*

(WMC) *If α is a pole of $Z_f(s)$ then $\exp(2\pi i\alpha)$ is a monodromy eigenvalue along $f = 0$.*

General pattern of known results: need luck for explicit resolution, other type of luck for $b_f(s)$.

Remark

No systematic approach known. Isolated singularity case is open.

For arrangements:

Theorem (Budur–Mustață–Teitler)

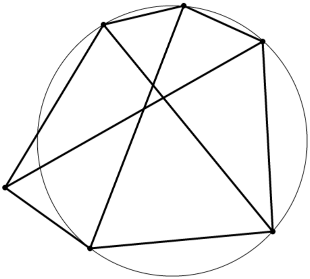
The zeta function of an arrangement is combinatorial.
(from Schechtman/Terao/Varchenko, de Concini/Procesi wonderful compactifications).

Example (-)

The Bernstein–Sato polynomial of an arrangement is not a function of the (traditional) intersection lattice.

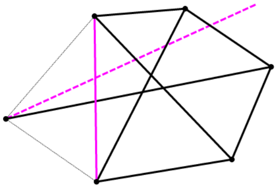
An example of b-functions

$S = 6$ points in $\mathbb{P}^2$; $f = 6$ sides, 3 main diagonals, no "accidents".



An example of a b-function, II

Generically, $H^0_{\mathfrak{m}}(R/J)$ is generated in degree 9 by 6 forms like this:



Dashed line:
“symmetry axis”.

One can force degree 8 elements in $H^0_{\mathfrak{m}}(R/J)$ that vanish along five of the arrangement lines.
Doing so forces the triple points to be on a quadric.

One can now link these elements of $H^0_{\mathfrak{m}}(R/J)$ to a new root.

Monodromy conjecture for arrangements

So, $Z_f(s)$ of an arrangement is combinatorial, but $b_f(s)$ is not.

Theorem

(WMC) holds for arrangements. (Budur–Mustață–Teitler)
(SMC) holds if L is nice (Budur–Saito–Yuzvinsky)

In fact:

Theorem (Budur–Mustață–Teitler)

If $b_f(-n/d) = 0$ for all indecomposable arrangements then (SMC) is true for all arrangements.

This *does* have combinatorial flavor.

Remark

There is an arrangement where n/d is not pole of zeta function (Veys). Yet, it has $-n/d$ as root, [-].

(SMC) and $\mathcal{L}_f$

Theorem (-)

Let f be an arrangement with $\operatorname{pdim}(\Omega^i(\log f)) \leq i$. *(So f is* tame*)*
Then (SMC) holds.

Steps:

$$[\text{tame}] \Leftrightarrow [\mathcal{L}_f \text{ prime}] \Rightarrow [\operatorname{ann}(f^s) \text{ from derivations}]$$

$$\Rightarrow [b_f(-n/d) = 0] \Rightarrow \text{(SMC)},$$

interspersed with a computation of $S = \operatorname{gr}_{0,1}(\operatorname{ann}(f^s))$.

Oddities

Remark

- $f = xyzw(y+w)(z+w)(x+y)(x+z)(x+z)(x+y+z+w)$ *is "one of only two" known non-tame arrangements.*
- *Even for those f, (SMC) and (n/d) holds...*
- *and so does Terao...*
- *BUT* $\operatorname{ann}(f^s)$ *not from derivations...*
- *"because" $\mathcal{L}_f$ has embedded component over 0.*

Problem:

link "non-derivations" in $\operatorname{ann}_{D[s]}(f^s)$ to non-tame phenomena.

(This might kill off (SMC) for arrangements,
BECAUSE
it suffices to show that all non-derivation generators of $\operatorname{ann}_{D[s]}(f^s)$ have in each term more "x" than "∂").

Logarithmic b-functions

General case: $b_f(s)$ is minimal monic in left-ideal

$$\mathbb{C}[s] \cap \operatorname{ann}_{D[s]}(f^s) + D[s] \cdot f$$

Problem:

Can one replace $\operatorname{ann}_{D[s]}(f^s)$ with smaller ideal?
For example, $D[s] \cdot (\operatorname{Der}(-\log f), E - ds)$?

An example

- Let $f = xyz(x+w)(y+w)(z+w)(x+y+w)(x+z+w)(y+z+w)$.
- $\mathrm{ann}_{D[s]}(f^s)/D[s]\cdot\mathrm{Der}(-\log f)$ generated by "long" operator P of order 2.
- (proof: characteristic ideal of $D[s]\cdot(P,\mathrm{Der}(-\log f))$)
- Considering lead terms: $(s+2/3)P \in D[s]\cdot\mathrm{Der}(-\log f)$.

Logarithmic b-functions of arrangements

Theorem

For every arrangement $\mathcal{A}$,

$$\mathbb{C}[s] \cap D_n[s] \cdot (f_{\mathcal{A}}, \mathrm{Der}(-\log f_{\mathcal{A}}))$$

is nontrivial.
In general: unknown.

Definition

For an arrangement $\mathcal{A}$, the monic minimal polynomial annihilating $\mathrm{ann}_{D[s]}(f^s)/D[s] \cdot \mathrm{Der}(-\log f)$ is $b_{\log \mathcal{A}}(s)$.

Open Questions

Problems

- What do the roots of $b_{\log \mathcal{A}}(s)$ mean?
- Does every divisor have a logarithmic b-function?
- Is $\operatorname{gr}_{(0,1)} \operatorname{ann}_D(f^s)$ always prime?
- Can -1 be a root of the logarithmic b-function?