Rank one local systems

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Differential and combinatorial aspects of singularities

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Conjecture:

Let $X$ be either one of the following:

(A) compact Kähler manifold

(B) small ball $f^{-1}(0)$, where $f$ is germ of holomorphic function on $\mathbb{C}^n$.

Then $\Sigma_i k(X) = \text{finite union of torsion translated subtori of } M_B(X)$.

Long list of partial cases: Beauville, Green-Lazarsfeld, Arapura, Simpson, Campana, Pink-Roessler, Dimca-Papadima-Suciu, Dimca-Papadima, etc.

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2. Results

\[ (1) \quad \bar{X} = \bar{X} \text{ (sncd)}, \quad \bar{X} = \text{compact Kähler mfd}, \]

\[ H^1(\bar{X}, \mathbb{C}) = 0 \implies \Sigma_i k(X) = \text{finite union of unitary translated subtori}. \]

\[ (2) \quad \text{Case (A) for positive dimensional components of } \Sigma_1(X). \]

\[ \text{[Budur - Wang 2013]}: \quad X = \text{smooth quasi-projective complex variety}. \]

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[Dimca-Papadima-Suciu 2009]:

\[X = \text{formal space} \Rightarrow TC_1(\Sigma_1^k(X)) = \{w \in H_1(X, \mathbb{C}) | \dim H_q(X, \mathbb{C}) \cup w \geq k \}\]

and the components of \(\Sigma_1^k(X)\) through 1 are subtori.

[Durfee-Hain 1988]: \(X = (\text{small ball}) \setminus f^{-1}(0)\), where \(f\) is germ of holomorphic function on \(\mathbb{C}^n \Rightarrow X = \text{formal}\).

Corollary: Case (B) for components of \(\Sigma_1^k(X)\) passing through 1.

Small ball complements (case (B)) are related to classical singularity theory.
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Small ball complements (case (B)) are related to classical singularity theory:
3. Relation with Milnor fibers

Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be germ of holomorphic function, \( F_f = \) Milnor fiber = (small ball) \( \setminus f^{-1}(t) \) for \( 0 < |t| \ll 1 \).

Monodromy Theorem: The set \( E(f) \) of eigenvalues of monodromy on \( H^q(F_f, \mathbb{C}) \) consists of roots of unity.

Proposition: \( X = (\text{small ball}) \setminus f^{-1}(0) \Rightarrow E(f) = (\text{diagonal}) \cap \Sigma(X) \).

Here the 1-diml diagonal is in \( MB(X) = (\mathbb{C}^*)^r \), \( r = \) number of branches of \( f \).

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Let \( F = (f_1, \ldots, f_r) \), \( f_j \in \mathbb{C}[x_1, \ldots, x_n] \), \( f = \prod_{j=1}^r f_j \), \( x \in f^{-1}(0) \),

\[ U_{F,x} = (\text{small ball at } x) \setminus f^{-1}(0). \]

Then \( \Sigma(U_{F,x})^{\text{unif}} \subset (\mathbb{C}^*)^r \) for all \( x \).

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\Sigma(F) := \bigcup_{x \in f^{-1}(0)} \Sigma(U_{F,x})^{\text{unif}} \subset (\mathbb{C}^*)^r
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[lichtin, sabbah 1987]: $B_F \neq 0$. 
Conjecture [B. 2013]:

\[ \text{Exp}(\text{Zero}(\mathcal{B} \mathcal{F})) = \sum(\mathcal{F}) \].

Theorem [B. 2013]:

\[ \text{Exp}(\text{Zero}(\mathcal{B} \mathcal{F})) \supset \sum(\mathcal{F}). \]

Sanity check:

\[ \sum(\mathcal{F}) = \text{combinatorial if } f_i \text{ are hyperplanes.} \]

Easy.

Eg: \[ \mathcal{F} = (x, y, x + y, z, x + y + z) \]. That is, cone over these lines:

Nero Budur (KU Leuven)  
Rank one local systems
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Then $\Sigma(F)$ in $(C^\infty)^5$ is given by
\[
(t_1 t_2 t_3 - 1)(t_3 t_4 t_5 - 1)(t_1 t_2 t_3 t_4 t_5 - 1) \prod_{5 \leq j \leq 1} (t_j - 1) = 0.
\]

Computations with RISA/ASIR and DMOD in SINGULAR coupled with new techniques give $B_F$, confirming the conjecture.

The conjecture relating $B_F$ with rank one local systems is our attempt to uncover the deeper reasons beyond the classical result: [Malgrange 1982, Kashiwara 1983]: Case $r = 1$ of conjecture is true.
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$$A^\bullet_{DR}(X, \mathbb{C}) \xrightarrow{\text{Morgan}} \left( Gys^\bullet(X) = \bigoplus_{|I|=\bullet-i} H^i(D_I, \mathbb{C}), d^\bullet \right).$$
Show that any irreducible component of $\Sigma_{i} k(X)$ contains a torsion point.

Using finite cyclic covers, the proof is then reduced to (i).

Theorem (B - W):
If $S \subset C^n$, $T \subset (C^*)^n$, both Zariski closed and defined over $\bar{\mathbb{Q}}$, $\dim S = \dim T$, $\text{Exp}(S) \subset T$.

Then $T = \text{torsion translate of subtorus}$. Upcoming:
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Nero Budur (KU Leuven) 

Rank one local systems
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Any two distinct irreducible components of $\Sigma^k(X)$ intersect in at most finitely many points (torsion).

Any point in the intersection of an irreducible component of $\Sigma^k(X)$ with a distinct irreducible component of $\Sigma^l(X)$ lies in $\Sigma^k(X) + \Sigma^l(X)$.

If $X$ is smooth projective, then $\text{codim} \Sigma^i(X) \geq 2(|i - n| - \delta(X))$, where $\delta(X)$ is the defect of semi-smallness of the Albanese map of $X$. 

Nero Budur (KU Leuven)  
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Nero Budur (KU Leuven)  
Rank one local systems
If $X$ = compact complex torus, $K \in \text{Perv}(X)$ coming from a polarizable real Hodge module, 

$$\Sigma_i^k(X, K) := \{ \rho \in M_{MB}(X) \mid \text{dim} \text{H}^i(X, K \otimes \mathbb{C}L) \geq k \},$$

then $\Sigma_i^k(X, K)$ is a finite union of translated subtori. This is a counterpart of an older result for affine complex tori:

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\text{If } X = (\mathbb{C}^*)^n, \quad K \in \mathcal{Perv}(X), \quad \text{then } \sum i(k)(X, K) \text{ is a finite union of translated subtori, and } \text{codim } \sum i(k)(X, K) \geq \text{some expression}.
\]

**Corollary [Loeser-Sabbah 1991]:**

If \( X(\mathcal{d}) \subset (\mathbb{C}^*)^n \) is a closed subvariety such that \( X[\mathcal{d}] = \text{a perverse sheaf} \) (e.g. \( X = \text{lci} \)), then \( (-1)^d \chi(X) \geq 0 \).

**[Huh-Sturmfels 2013]:**

Conjectured same for any closed subvariety \( X \subset (\mathbb{C}^*)^n \).

Inspired by another conjecture:

\[
\text{MLdeg}(X) \geq (-1)^d \chi(X).
\]

**MLdeg:** the number of critical points of the likelihood function \( l_\alpha(x) = \prod_{n=1}^x \alpha_i^i \) for random data \( \alpha \in \mathbb{C}^r \).
[Gabber-Loeser 1996]: If $X = (\mathbb{C}^*)^n$, 

$K \in Perv\left(X, (\mathbb{C}^*)^n\right)$, then

$\sum_{i} k_{i}(X, K)$ is a finite union of translated subtori,

and $\text{codim} \sum_{i} k_{i}(X, K) \geq i$.

Corollary [Loeser-Sabbah 1991]: If $X(d) \subset (\mathbb{C}^*)^n$ closed subvariety such that $C_X[d] = \text{perverse sheaf}$ (e.g. $X = \text{lci}$), then

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$\text{MLdeg}(X)$ of a statistical model $X$ is the number of critical points of the likelihood function $l_{\alpha}(x) = \prod_{i=1}^{n} x_{\alpha_i}^{i}$ for random data $\alpha \in \mathbb{C}^r$. 
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Nero Budur (KU Leuven) 

Rank one local systems
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[Budur - Wang 2014]: Counterexamples to both conjectures.

Moreover, $\text{MLdeg}(X) \geq \left( -1 \right)^d \chi(X)$ for all closed subvarieties $X \subset (\mathbb{C}^*)^n$.

Note: from the result of Gabber-Loeser, one has $\left( -1 \right)^d \chi(X) \geq 0$.

So intersection cohomology and perverse sheaves theory of Goresky and MacPherson is relevant to statistics!

Thank you for your attention!
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