Rank one local systems

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Differential and combinatorial aspects of singularities

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Case (B) was wrongly claimed by Libgober.

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Small ball complements (case (B)) are related to classical singularity theory:

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Here the 1-diml diagonal is in $M_B(X) = (\mathbb{C}^*)^r$, $r =$ number of branches of f. So case (B) is ok for slices of $\Sigma^i_k(X)$ by 1-dim algebraic subgroups.

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for some $P \in \mathbb{C}[x, \partial/\partial x, s]$.

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$$\Sigma(F) := \bigcup_{x\in f^{-1}(0)} \Sigma(U_{F,x})^{unif} \subset (\mathbb{C}^*)^r$$

= uniform cohomology support locus

$B_F :=$ the (Bernstein-Sato) ideal in $\mathbb{C}[s_1, \dots, s_r]$ generated by b with

$$b(s_1, \dots, s_r) f_1^{s_1} \dots f_r^{s_r} = P f_1^{s_1+1} \dots f_r^{s_r+1}$$

for some $P \in \mathbb{C}[x, \partial/\partial x, s]$.

[Lichtin, Sabbah 1987]: $B_F \neq 0$.

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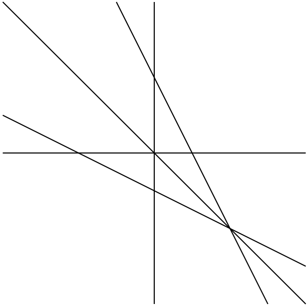

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[Malgrange 1982, Kashiwara 1983]: Case $r=1$ of conjecture is true.

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Upcoming: We use this to prove case (B) of Conjecture on $\Sigma^i_k(X)$.

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[Dimca 2007]: Any irreducible comp of $\Sigma^1_k(X)$ is also one for $\Sigma^1_l(X)$ with $l \leq k$.

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This is a counterpart of an older result for affine complex tori:

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$MLdeg(X)$ of a statistical model X is the number of critical points of the likelihood function $l_\alpha(x) = \prod_{i=1}^n x_i^{\alpha_i}$ for random data $\alpha \in \mathbb{C}^r$.

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Thank you for your attention !