Resonance varieties and Chen ranks of braid-like groups

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Overview

1. Rational homotopy theory
2. Formality properties
3. Resonance varieties and Chen ranks
4. Braid-like groups

Minimal models

- $\Bbbk$: a field of characteristic 0.
- (A^*, d): a commutative differential graded algebra over $\Bbbk$ (DGA).
- A *Hirsch extension* (of degree j) is a DGA inclusion $\alpha\colon (A^*, d_A) \hookrightarrow (A^* \otimes \bigwedge(V), d)$, with $\deg(V) = j$ and $d(V) \subset A^{j+1}$.
- (A^*, d) is *minimal* if $A^0 = \Bbbk$, and satisfying:
  1. $A^* = \bigcup_{j \geq 0} A^*_j$, where A_j is a Hirsch extension of A_{j-1}.
  2. d is *decomposable*, i.e., $dA^* \subset A^+ \wedge A^+$, where $A^+ = \bigoplus_{i \geq 1} A^i$.
- A DGA morphism $f\colon A \to B$ is an *i-quasi-isomorphism* if $f^*\colon H^j(A) \to H^j(B)$ is an isomorphism for each $j \leq i$ and monomorphism for $j = i+1$.
- A and B are *i-weakly equivalent* ($A \simeq_i B$) if there is a zig-zag of i-quasi-isomorphism connecting A to B.
- If B is a minimal DGA generated by elements of degree $\leq i$, and there exists an i-quasi-isomorphism $f\colon B \to A$, then we say that B is an *i-minimal model* for A.

• Each connected DGA A has an i-minimal model $\mathcal{M}(A, i)$, unique up to isomorphism. (Sullivan 77, Morgan 78)

Formality Properties

- (A^*, d) is said to be *i-formal* if there exists an i-quasi-isomorphism $\mathcal{M}(A, i) \to (H^*(A), 0)$. Equivalently, $(A^*, d) \simeq_i (H^*(A), 0)$.
- $A_{PL}(X)$: the rational Sullivan model of a connected space X.
- X is said to be *(i-)formal*, if $A_{PL}(X)$ is (i-)formal.
- Every i-formal space X with $H^{\geq i+2}(X; \mathbb{Q}) = 0$ is formal. (Măcinic 10)

Theorem (Sullivan 77, Neisendorfer–Miller 78, Halperin–Stasheff 79)

Let $\mathbb{Q} \subset \mathbb{k}$ be a field extension, and X be a connected space with finite Betti numbers. X is formal over $\mathbb{Q}$ if and only if X is formal over $\mathbb{k}$.

Theorem (Suciu-W. 15)

Let X be a connected space with finite Betti numbers $b_1(X), \dots, b_{i+1}(X)$. Then X is i-formal over $\mathbb{Q}$ if and only if X is i-formal over $\mathbb{k}$.

- The 1-formality of a path-connected space X depends only on $\pi_1(X)$.
- A finitely generated group G is called *1-formal*, if $X = K(G, 1)$ is 1-formal, i.e., $\mathcal{M}(X, 1)$ is 1-quasi-isomorphic to $(H^*(G; \mathbb{Q}), 0)$.

Malcev Lie algebras

- G: a finitely generated group.
- The *lower central series* of G: $\Gamma_1 G = G$, $\Gamma_{k+1} G = [\Gamma_k G, G]$, $k \geq 1$.
- From the tower $(G/\Gamma_2 G)\otimes\Bbbk \leftarrow (G/\Gamma_3 G)\otimes\Bbbk \leftarrow (G/\Gamma_4 G)\otimes\Bbbk \leftarrow \cdots$, we get a tower of nilpotent Lie algebras

$$\mathfrak{L}((G/\Gamma_2 G)\otimes\Bbbk) \twoheadleftarrow \mathfrak{L}((G/\Gamma_3 G)\otimes\Bbbk) \twoheadleftarrow \mathfrak{L}((G/\Gamma_4 G)\otimes\Bbbk) \twoheadleftarrow$$

- Let $\mathcal{M}(G,1)$ be the 1-minimal model of $K(G,1)$ (over $\Bbbk$). Take the dual of $\mathcal{M}(G,1)^1_1 \subset \mathcal{M}(G,1)^1_2 \subset \cdots \subset \mathcal{M}(G,1)^1_j \subset \cdots$, we also get a tower of nilpotent Lie algebras

$$\mathfrak{L}_1(G) \twoheadleftarrow \mathfrak{L}_2(G) \twoheadleftarrow \cdots \twoheadleftarrow \mathfrak{L}_j(G) \twoheadleftarrow \cdots .$$

- These two towers of nilpotent Lie algebras are isomorphic.(Sullivan 77, Cenkl–Porter 81).
- The inverse limit of the tower is called the *Malcev Lie algebra* of G (over $\Bbbk$), denoted by $\mathfrak{m}(G;\Bbbk)$.

Example

Let X be the Heisenberg manifold, and $G = \pi_1(X)$.

- The (1-)minimal model $\mathcal{M}(X)$ is $\bigwedge(a, b, c)$ with $d(a) = d(b) = 0$ and $d(c) = a \wedge b$.
- $$\begin{array}{ccc} \mathcal{M}(G)_1^1 & \hookrightarrow & \mathcal{M}(G)_2^1 \\ \| & & \| \\ \Bbbk^2\{a, b\} & & \Bbbk^3\{a, b, c\} \end{array} \qquad \begin{array}{ccc} \mathfrak{L}_1(G) & \twoheadleftarrow & \mathfrak{L}_2(G) \\ \| & & \| \\ \Bbbk^2\{a^*, b^*\} & & \Bbbk^3\{a^*, b^*, c^*\} \end{array}$$
  $d(a) = d(b) = 0$ $\qquad$ $[a^*, b^*] = c^*$

  $d(c) = a \wedge b$ $\qquad$ $[a^*, c^*] = [b^*, c^*] = 0.$
- The Malcev Lie algebra $\mathfrak{m}(G; \Bbbk) \cong \mathrm{Lie}(x, y)/\Gamma_3 \mathrm{Lie}(x, y)$.
- G is not 1-formal. (Non-vanishing Massey products)

Graded Lie algebras

- G : a finitely generated group.
- The *associated graded Lie algebra* of a group G is defined by

$$\mathrm{gr}(G;\Bbbk) := \bigoplus_{k\geq 1}(\Gamma_k G/\Gamma_{k+1} G)\otimes_{\mathbb{Z}}\Bbbk.$$

- The *holonomy Lie algebra* of a group G is defined to be

$$\mathfrak{h}(G;\Bbbk) := \mathrm{Lie}(H_1(G;\Bbbk))/\langle \mathrm{im}(\partial_G)\rangle.$$

Here, ∂_G is the dual of $H^1(G;\Bbbk)\wedge H^1(G;\Bbbk) \xrightarrow{\cup} H^2(G;\Bbbk)$.

Proposition

- *There exists an epimorphism* $\Phi_G\colon \mathfrak{h}(G;\Bbbk) \twoheadrightarrow \mathrm{gr}(G;\Bbbk)$. [Lambe 86]
- $\mathrm{gr}(G;\Bbbk) \xrightarrow{\cong} \mathrm{gr}(\mathfrak{m}(G;\Bbbk))$. [Quillen 68]

Partial Formality of groups

- A group G is 1-formal iff $\mathfrak{m}(G;\Bbbk) \cong \widehat{\mathfrak{h}}(G;\Bbbk)$. [Markl–Papadima92]
- A group G is called *graded-formal*, if $\Phi_G\colon \mathfrak{h}(G;\Bbbk) \twoheadrightarrow \operatorname{gr}(G;\Bbbk)$ is an isomorphism of graded Lie algebras.
- A group G is called *filtered-formal*, if there is a filtered Lie algebra isomorphism $\mathfrak{m}(G;\Bbbk) \cong \widehat{\operatorname{gr}}(G;\Bbbk)$.

$$\begin{array}{ccc}
\mathfrak{m}(G;\Bbbk) & \xrightarrow{\ 1-\text{formal}\ } & \widehat{\mathfrak{h}}(G;\Bbbk) \\
{\scriptstyle \text{filtered}-\text{formal}} \searrow & & \swarrow {\scriptstyle \text{graded}-\text{formal}} \\
& \widehat{\operatorname{gr}}(\mathfrak{m}(G)) \cong \widehat{\operatorname{gr}}(G;\Bbbk). &
\end{array}$$

Theorem (Suciu–W.15)

A finitely generated group G is filtered-formal (graded-formal) over $\mathbb{Q}$ if and only it is filtered-formal (graded-formal) over $\Bbbk$.

Remark

- formal $\Longrightarrow$ i-formal $\Longrightarrow$ 1-formal $\Longleftrightarrow$ $\begin{array}{c}\text{graded-formal}\\ +\\ \text{filtered-formal.}\end{array}$

Example

Let X be the Heisenberg manifold, and $G = \pi_1(X) \cong F_2/\Gamma_3 F_2$.

- The Malcev Lie algebra $\mathfrak{m}(G;\Bbbk) \cong \mathrm{Lie}(x,y)/\Gamma_3 \mathrm{Lie}(x,y) \cong \mathrm{gr}(G;\Bbbk)$.
- The holonomy Lie algebra is $\mathfrak{h}(G;\Bbbk) = \mathrm{Lie}(x,y)$.
- G is filtered-formal, but not graded-formal.

- The filtered formality of finite-dimensional, nilpotent Lie algebras has many different names: 'Carnot', 'naturally graded', 'homogeneous' and 'quasi-cyclic')
- The complement of a chordal graphic arrangement of a complex projective curve of $g > 0$ is not 1-formal in general, but always filtered-formal. [Bezrukavnikov, Bibby, ...]

Seifert fibered manifolds

Let $\eta\colon M \to \Sigma_g$ be an orientable Seifert fibered manifold with Seifert invariants $(g, b, (\alpha_i, \beta_i), i = 1, \dots, s)$. Let $e(\eta)$ be its Euler number.

Theorem (Putinar 98)

If $g > 0$, the minimal model $\mathcal{M}(M)$ is the Hirsch extension $\mathcal{M}(\Sigma_g) \otimes (\bigwedge(c), d)$, with differential $d(c) = 0$ if $e(\eta) = 0$, and $d(c) \in \mathcal{M}^2(\Sigma_g)$ represents a generator of $H^2(\Sigma_g; \Bbbk)$ if $e(\eta) \neq 0$.

Proposition

The Malcev Lie algebra of $\pi_\eta := \pi_1(M)$ is the degree completion of the graded Lie algebra $L(\pi_\eta) =$

$$\begin{cases} \mathrm{Lie}(x_1, y_1, \dots, x_g, y_g, z)/\langle \sum_{i=1}^{g} [x_i, y_i] = 0,\ z \text{ central}\rangle & \text{if } e(\eta) = 0; \\ \mathrm{Lie}(x_1, y_1, \dots, x_g, y_g, w)/\langle \sum_{i=1}^{g} [x_i, y_i] = w,\ w \text{ central}\rangle & \text{if } e(\eta) \neq 0, \end{cases}$$

where $\deg(w) = 2$ and the other generators have degree 1.
Moreover, $\mathrm{gr}(\pi_\eta; \Bbbk) \cong L(\pi_\eta)$.

Proposition (Suciu–W. 15)

Let $\eta\colon M \to \Sigma_g$ be a Seifert fibration. The rational holonomy Lie algebra of the group $\pi_\eta = \pi_1(M)$ is given by $\mathfrak{h}(\pi_\eta; \Bbbk) =$

$$\begin{cases} \mathrm{Lie}(x_1, y_1, \dots, x_g, y_g, h)/\langle \sum_{i=1}^{s}[x_i, y_i] = 0, h \text{ central}\rangle & \text{if } e(\eta) = 0; \\ \mathrm{Lie}(2g) & \text{if } e(\eta) \neq 0. \end{cases}$$

Corollary

Fundamental groups of orientable Seifert manifolds are filtered-formal.

Corollary

If $g = 0$, the group π_η is always 1-formal, while if $g > 0$, the group π_η is graded-formal if and only if $e(\eta) = 0$.

Propagation of partial formalities

Proposition (Suciu–W.15)

Let G be a finitely generated group, and let $K \leq G$ be a subgroup. Suppose there is a split monomorphism $\iota\colon K \to G$. Then:

1. *If G is graded-formal, then K is also graded-formal.*
2. *If G is filtered-formal, then K is also filtered-formal.*
3. *If G is 1-formal, then K is also 1-formal.*

Proposition (Suciu–W.15)

Let G_1 and G_2 be two finitely generated groups. The following conditions are equivalent.

1. *G_1 and G_2 are graded-formal (respectively, filtered-formal, or 1-formal).*
2. *$G_1 * G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).*
3. *$G_1 \times G_2$ is graded-formal (respectively, filtered-formal, or 1-formal).*

Resonance varieties

- Suppose $A := H^*(G, \mathbb{C})$ has finite dimension in each degree.
- For each $a \in A^1$, we have $a^2 = 0$.
- Define a cochain complex of finite-dimensional $\mathbb{C}$-vector spaces,

$$(A, a) : A^0 \xrightarrow{a\cup -} A^1 \xrightarrow{a\cup -} A^2 \xrightarrow{a\cup -} \cdots ,$$

  with differentials given by left-multiplication by a.
- The *resonance varieties* of G are the homogeneous subvarieties of A^1

$$\mathcal{R}_i(G, \mathbb{C}) = \{a \in A^1 \mid \dim_{\mathbb{C}} H^1(A; a) \geq i\}.$$

- $\mathcal{R}_1(\mathbb{Z}^n, \mathbb{C}) = \{0\}$; $\mathcal{R}_1(\pi_1(\Sigma_g), \mathbb{C}) = \mathbb{C}^{2g},\ g \geq 2$.

Theorem (Dimca, Papadima, Suciu 09)

If G is 1-formal, then $\mathcal{R}^1_d(G, \mathbb{C})$ is a union of rationally defined linear subspaces of $H^1(G, \mathbb{C})$.

Chen Lie algebras

- The *Chen Lie algebra* of a finitely generated group G is defined to be

$$\mathrm{gr}(G/G''; \Bbbk) := \bigoplus_{k\geq 1} (\Gamma_k(G/G'')/\Gamma_{k+1}(G/G'')) \otimes_{\mathbb{Z}} \Bbbk.$$

- The quotient map $h\colon G \twoheadrightarrow G/G''$ induces $\mathrm{gr}(G; \Bbbk) \twoheadrightarrow \mathrm{gr}(G/G''; \Bbbk)$.
- The *LCS ranks* of G are defined as $\phi_k(G) := \mathrm{rank}(\mathrm{gr}_k(G; \Bbbk))$.
- The *Chen ranks* of G are defined as $\theta_k(G) := \mathrm{rank}(\mathrm{gr}_k(G/G''; \Bbbk))$.
- $\theta_k(G) = \phi_k(G)$ for $k \leq 3$.
- $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$, $k \geq 2$. [Chen 51]

Theorem (Labute 08, Suciu-W. 15)

For each $i \geq 2$, the quotient map $G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded $\Bbbk$-Lie algebras,

$$\Psi_G^{(i)} \colon \operatorname{gr}(G;\Bbbk)/\operatorname{gr}(G;\Bbbk)^{(i)} \longrightarrow\!\!\!\!\rightarrow \operatorname{gr}(G/G^{(i)};\Bbbk)\,.$$

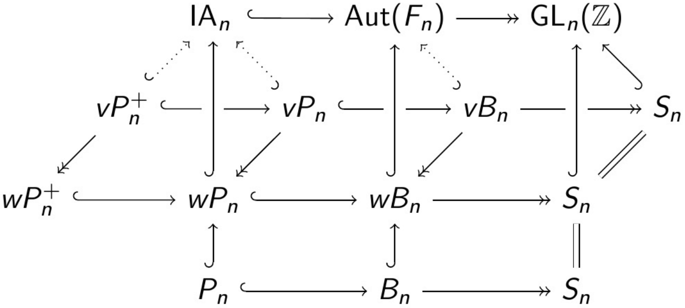
Moreover, if G is a filtered-formal group, then each solvable quotient $G/G^{(i)}$ is also filtered-formal, and the map $\Psi_G^{(i)}$ is an isomorphism.

Corollary (Papadima-Suciu 04)

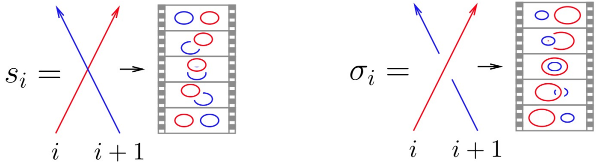
If G is a 1-formal group, then $\mathfrak{h}(G;\Bbbk)/\mathfrak{h}(G;\Bbbk)^{(i)} \cong \operatorname{gr}(G/G^{(i)};\Bbbk)$.

In particular, for $i = 2$, the above isomorphisms provide alternative (easier) methods to compute the Chen ranks.

Braid-like groups



Generators for welded braid groups wB_n(also for virtual braid groups vB_n):



(Pictures from Bar-Natan and Dancso)

Relations for welded braid groups wB_n:

$$\begin{cases} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & i = 1, 2, \ldots, n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| \geq 2. \end{cases} \tag{1}$$

$$\begin{cases} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & i = 1, 2, \ldots, n-2, \\ s_i s_j = s_j s_i, & |i-j| \geq 2. \\ s_i^2 = 1, & i = 1, 2, \ldots, n-1; \end{cases} \tag{2}$$

$$\begin{cases} s_i\sigma_j = \sigma_j s_i, & |i-j| \geq 2, \\ \sigma_i s_{i+1} s_i = s_{i+1} s_i \sigma_{i+1}, & i = 1, 2, \ldots, n-2, \\ s_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i s_{i+1}, & i = 1, 2, \ldots, n-2. \end{cases} \tag{3}$$

- The presentation of wP_n was given by McCool (86).
  This is the same as the fundamental group of the untwisted flying rings space given by Brendle and Hatcher (13).
- The presentation of vP_n was given by Bardakov (04).
- Both vP_n and wP_n have generators x_{ij} for $i \neq j$ and both have subgroups generated by x_{ij} for $i < j$ denoted by vP_n^+ and wP_n^+.

Pure virtual braid groups

Theorem (Suciu, W. 15)

The pure virtual braid groups vP_n and vP_n^+ are 1-formal if and only if $n \leq 3$.

Sketch of proof:

Lemma

There are split monomorphisms

$$\begin{array}{ccccccccccc}
vP_2^+ & \hookrightarrow & vP_3^+ & \hookrightarrow & vP_4^+ & \hookrightarrow & vP_5^+ & \hookrightarrow & vP_6^+ & \hookrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
vP_2 & \hookrightarrow & vP_3 & \hookrightarrow & vP_4 & \hookrightarrow & vP_5 & \hookrightarrow & vP_6 & \hookrightarrow & \dots
\end{array}$$

Lemma

The group vP_3 is 1-formal.

Proof: $vP_3 \cong N * \mathbb{Z}$, and $P_4 \cong N \times \mathbb{Z}$.
Next we show that vP_4^+ is not 1-formal.

Lemma

The first resonance variety $\mathcal{R}_1(vP_4^+, \mathbb{C})$ is the subvariety of $\mathbb{C}^6$ given by the equations

$$x_{12}x_{24}(x_{13} + x_{23}) + x_{13}x_{34}(x_{12} - x_{23}) - x_{24}x_{34}(x_{12} + x_{13}) = 0,$$
$$x_{12}x_{23}(x_{14} + x_{24}) + x_{12}x_{34}(x_{23} - x_{14}) + x_{14}x_{34}(x_{23} + x_{24}) = 0,$$
$$x_{13}x_{23}(x_{14} + x_{24}) + x_{14}x_{24}(x_{13} + x_{23}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) = 0,$$
$$x_{12}(x_{13}x_{14} - x_{23}x_{24}) + x_{34}(x_{13}x_{23} - x_{14}x_{24}) = 0.$$

$\Rightarrow$ The group vP_4^+ is not 1-formal. $\Rightarrow$ vP_n^+ is not 1-formal for $n \geq 4$.

Remark

The cohomology algebras $H^*(vP_n;\mathbb{C})$ and $H^*(vP_n^+;\mathbb{C})$ were computed by Bartholdi, Enriquez, Etingof, and Rains 06, and Lee 13. They also showed that vP_n and vP_n^+ are graded-formal.

Pure welded braid groups wP_n

- wP_n and wP_n^+ are 1-formal.[Berceanu-Papadima 09]
- $H^*(wP_n, \mathbb{C})$ was computed by Jensen, McCammond, and Meier (06).
- D.Cohen (09) computed the first resonance variety of the group wP_n:

$$\mathcal{R}_1(wP_n, \mathbb{C}) = \bigcup_{1 \le i < j \le n} C_{ij} \cup \bigcup_{1 \le i < j < k \le n} C_{ijk},$$

  where $C_{ij} = \mathbb{C}^2$ and $C_{ijk} = \mathbb{C}^3$.
- D.Cohen and Schenck (13) showed that the Chen ranks of wP_n are given by the 'Chen ranks formula'

$$\theta_k(wP_n) = (k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}$$

  for k large enough.

Upper pure welded braid groups wP_n^+

- F. Cohen, Pakhianathan, Vershinin, and Wu (07) computed the cohomology ring $H^*(wP_n^+;\mathbb{Z})$.
- The LCS ranks $\phi_k(wP_n^+) = \phi_k(P_n)$.
  The Betti numbers $b_k(wP_n^+) = b_k(P_n)$.
- They ask a question: are wP_n^+ and P_n isomorphic for $n \geq 4$?
- For $n = 4$, the question was answered by Bardakov and Mikhailov (08) using Alexander polynomials.

Theorem (Suciu, W. 15)

The Chen ranks θ_k of wP_n^+ are given by $\theta_1 = \binom{n}{2}, \theta_2 = \binom{n}{3}, \theta_3 = 2\binom{n+1}{4}$,

$$\theta_k = \binom{n+k-2}{k+1} + \theta_{k-1} = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}, \ k \geq 4.$$

Corollary

The pure braid group P_n, the upper pure welded braid groups wP_n^+, and the product group $\Pi_n := \prod_{i=1}^{n-1} F_i$ are not isomorphic for $n \geq 4$.

Proof:

$$\theta_4(P\Sigma_n^+) = 2\binom{n+1}{4} + \binom{n+2}{5}, \theta_4(P_n) = 3\binom{n+1}{4}, \theta_4(\Pi_n) = 3\binom{n+2}{5}.$$

The Chen ranks of P_n and Π_n were computed by D. Cohen and Suciu (95).

Theorem (Suciu, W. 15)

The first resonance variety of the upper pure welded braid groups wP_n^+ is

$$\mathcal{R}_1(wP_n^+, \mathbb{C}) = \bigcup_{n \geq i > j \geq 2} C_{i,j},$$

where $C_{i,j} = \mathbb{C}^j$.

Remark (Chen ranks conjecture, Suciu 01, Schenck-Suciu 04, D. Cohen-Schenck 14)

Let c_n be the number of n-dimensional components of $\mathcal{R}_1(G)$.

$$\theta_k(G) = \sum_{n\geq 2} c_m \cdot \theta_k(F_n), \quad \text{for} \quad k \gg 1.$$

This formula is true if G is a 1-formal, commutator-relators group, such that the resonance variety $\mathcal{R}^1(G)$ is 0-isotropic, projectively disjoint, and reduced as a scheme.

Examples satisfying these conditions include hyperplane arrangement groups and pure welded braid group wP_n. However, wP_n^+ do not satisfy this formula for $n \geq 4$.

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Thank You!