MATROID CONNECTIVITY AND SINGULARITIES OF CONFIGURATION HYPERSURFACES

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Abstract. Consider a linear realization of a matroid over a field. One associates to it a configuration polynomial and bilinear form with polynomial coefficients. The corresponding configuration hypersurface and its non-smooth locus support the respective first and second degeneracy scheme of the bilinear form.

We describe the effect of matroid connectivity on these schemes: For (2-)connected matroids, the configuration hypersurface is integral, and the second degeneracy scheme is reduced Cohen–Macaulay of codimension 3. If the matroid is 3-connected, then also the second degeneracy scheme is integral.

In the process, we describe the behavior of configuration polynomials, forms and schemes with respect to various matroid constructions.

Contents

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1. Introduction

1.1. **Feynman diagrams.** A fundamental problem in high-energy physics is to understand the scattering of particles. The basic tool for theoretical predictions is the Feynman diagram with underlying Feynman graph $G = (V, E)$. The scattering data correspond to Feynman amplitudes, integrals computed in the positive orthant of the projective space labeled by the internal edges of the Feynman graph. The integrand is a rational function in the edge variables x_e, $e \in E$, that depends parametrically on the masses and moments of the involved particles (see [Bro17]).

The convergence of a Feynman amplitude is determined by the structure of the denominator, which in any case involves (a power of) the *Symanzik polynomial* $\sum_T \prod_{e \notin T} x_e$ of G where T runs through the spanning trees of G. For graphs with edge number less than twice the loop number the denominator also involves (a power of) the second Symanzik polynomial obtained by summing over 2-forests and involves masses and moments. Symanzik polynomials can factor, and the singularities and intersections of the individual components determine the convergence of the Feynman amplitudes.

Remarkably, amplitudes tend to involve values of the Riemann zeta function, or more generally multiple zeta values and polylogarithms. In [BK97], Broadhurst and Kreimer display a large body of computational evidence that in the last to decades has become ever more impressive. Viewing amplitudes as periods, Kontsevich speculated that Symanzik polynomials, or equivalently their cousins the *Kirchhoff polynomials*

$$\psi_G(x) = \sum_T \prod_{e \in T} x_e,$$

with the sum again taken over the spanning trees of G, be mixed Tate; this would imply the relation to multiple zeta values. However, Belkale and Brosnan [BB03] proved that the collection of Kirchhoff polynomials is a rather complicated class of singularities: in finite characteristic, the counting function on the affine complements cannot always be polynomial in the size of the field. This does not exactly rule out that Feynman amplitudes are well-behaved, but makes it rather more unlikely. On the other hand, it makes the study of these singularities, and especially any kind of uniformity results, that much more interesting.

The influential paper [BEK06] of Bloch, Esnault and Kreimer generated a significant amount of work from the point of view of complex geometry: we refer to the book [Mar10] of Marcolli for exposition, as well as [Bro17; Dor11; BW10]. Varying ideas of Connes and Kreimer on renormalization that view Feynman integrals as specializations of the Tutte polynomial, Aluffi and Marcolli formulate in [AM11b; AM11a] parametric Feynman integrals as periods, leading to motivic studies on cohomology. On the explicit side, there is a large body of publications in which specific graphs and their polynomials and amplitudes are discussed. But, as Brown writes in [Bro15], while a diversity of techniques is used to study Feynman diagrams, "each new loop order involves mathematical objects which are an order of magnitude more complex than the last, [...] the unavoidable fact is that arbitrary amplitudes remain out of reach as ever."

The present article can be seen as the first step towards a search for uniform properties in this zoo of singularities. We view it as a stepping stone for further studies of invariants such as log canonical threshold, logarithmic differential forms and embedded resolution of singularities.

1.2. **Configuration polynomials.** The main idea of Belkale and Brosnan is to move the burden of proof into the more general realm of polynomials and constructible sets derived from matroids rather than graphs, and then to reduce to known facts about such polynomials. The article [BEK06] casts Kirchhoff and Symanzik polynomials as very special instances of *configuration polynomials*; this idea was further developed in [Pat10]. We consider this as a more natural setting since notions such as duality and truncation behave well for configuration polynomials as a whole, but these operations do not preserve the subfamily of matroids derived from graphs. In particular, we can focus exclusively on Kirchhoff/configuration polynomials, since the Symanzik polynomial of G appears as the configuration polynomial of the dual configuration induced by the incidence matrix of G.

The configuration polynomial does not depend on a matroid itself but on a configuration, that is, on a linear realization of a matroid over a field $\mathbb{K}$. The same matroid can admit different realizations, which, in turn, give rise to different configuration polynomials (see Example 5.3). The *matroid (basis) polynomial* is a competing object, which is assigned to any, even non-realizable, matroid. It has proven useful for combinatorial applications (see [AGV18; Piq19]). For graphs and, more generally, regular matroids, all configuration polynomials essentially agree with the matroid polynomial. However, they are different in general (see Example 5.2).

Configuration polynomials have better geometric properties than matroid polynomials: Generalizing the matrix-tree theorem, the configuration polynomial arises as the determinant of a symmetric bilinear *configuration form* with linear polynomial coefficients. As a consequence, the corresponding *configuration hypersurface* maps naturally to the generic symmetric determinantal variety. In the present article, we establish further uniform, geometric properties of configuration polynomials, which do not hold for matroid polynomials in general.

1.3. **Summary of results.** Some indication of what is to come can be gleaned from the following note by Marcolli in [Mar10, p. 71]: "graph hypersurfaces tend to have singularity loci of small codimension".

Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M on a set E. Fix coordinates $x_E = (x_e)_{e\in E}$. There is an associated *configuration polynomial* $\psi_W \in \mathbb{K}[x_E]$ and *configuration (bilinear) form* Q_W (see Definitions 3.2 and 3.21). They are related by $\psi_W = \det Q_W$ (see Lemma 3.24). The *configuration hypersurface* $X_W \subseteq \mathbb{K}^E$ defined by ψ_W can thus be seen as the *first degeneracy scheme* of Q_W (see Definition 4.7). The *second degeneracy scheme* $\Delta_W \subseteq \mathbb{K}^E$ defined by the submaximal minors of Q_W is a subscheme of the Jacobian scheme $\Sigma_W \subseteq \mathbb{K}^E$ of X_W defined by the partial derivatives of ψ_W (see Lemma 4.10). The latter defines the non-smooth locus of X_W (see Remark 4.8). Patterson showed Σ_W and Δ_W have the same underlying reduced scheme (see Theorem 4.13), that is,

$$\Delta_W \subseteq \Sigma_W \subseteq \mathbb{K}^E, \quad \Sigma_W^{\mathrm{red}} = \Delta_W^{\mathrm{red}}.$$

He mentions that he does not know the reduced scheme structure (see [Pat10, p. 696]). We show that Σ_W is not reduced in general (see Example 5.1), whereas Δ_W often is. Our main results from Theorems 4.23, 4.34 and 4.37 can be summarized as follows.

Main Theorem. *Let M be a connected matroid of rank $\operatorname{rk}\mathsf{M} \geqslant 2$ on the set E with a linear realization $W \subseteq \mathbb{K}^E$ over a field $\mathbb{K}$. Then $\Delta_W = \Sigma_W^{\mathrm{red}}$ is the non-smooth locus of X_W over $\mathbb{K}$. It is Cohen–Macaulay of codimension 3 in $\mathbb{K}^E$. Unless $\mathbb{K}$ has characteristic 2, Σ_W is generically reduced. If M is 3-connected, then Δ_W is integral and Σ_W is irreducible.* □

In case $\operatorname{rk}\mathsf{M} = 1$ the polynomial ψ_W is linear and hence $\Delta_W = \Sigma_W = \varnothing$ (see Remark 4.11.(a)). If M arises from adding (co)loops to a connected matroid, then reducedness of Δ_W persists (see Corollary 4.35). However if M is disconnected even when loops are removed, then Σ_W and hence Δ_W has codimension 2 in $\mathbb{K}^E$ (see Remark 4.9).

While our main objective is to establish the results above, along the way we continue the systematic study of configuration polynomials in the spirit of [BEK06; Pat10]. For instance, we describe the behavior of configuration polynomials with respect to connectedness, duality, deletion/contraction and 2-separations (see Propositions 3.10, 3.12, 3.14 and 3.27). Patterson showed that the *second Symanzik polynomial* associated with a Feynman graph is, in fact, a configuration polynomial: we note that the underlying matroid is a truncation of the circuit matroid of the graph, parameterized by the momentum parameters (see Proposition 3.20).

1.4. **Outline of the proof.** The proof of the Main Theorem intertwines methods from matroid theory, commutative algebra and algebraic geometry. In order to keep our arguments self-contained and accessible, we recall preliminaries from each of these subjects and give detailed proofs (see §2.1, §2.3 and §4.1).

An important commutative algebra ingredient is a result of Kutz (see [Kut74]). It bounds the grade of an ideal of submaximal minors of a symmetric matrix by 3 and yields perfection in case of equality. Kutz' result applies to the defining ideal of Δ_W. The codimension of Δ_W in $\mathbb{K}^E$ is therefore bounded by 3 and Δ_W is Cohen–Macaulay in case of equality (see Proposition 4.16). In particular Δ_W is pure-dimensional and hence reduced if generically reduced. Due to Patterson's result Σ_W is equidimensional in this case.

On the matroid side our approach makes use of *handles* (see Definition 2.2), which are called *ears* in case of graphic matroids. A *handle decomposition* builds up any connected matroid from a circuit by successively attaching handles (see Proposition 2.5). Conversely this yields for any connected matroid which is not a circuit a *non-disconnective* handle which leaves the matroid connected when deleted (see Definition 2.2). This allows one to prove statements on connected matroids by induction.

We describe the effect of deletion and contraction of a handle H to the configuration polynomial (see Corollary 3.15). In case the Jacobian scheme $\Sigma_{W\setminus H}$ associated with the deletion $\mathsf{M}\setminus H$ has codimension at least 3 we prove the same for Σ_W (see Lemma 4.21). Applied to a non-disconnective H it follows with Patterson's result that Δ_W reaches the dimension bound and is thus Cohen–Macaulay of codimension 3 (see Theorem 4.23). We further identify 3 (more or less explicit) types of generic points with respect to a non-disconnective handle (see Corollary 4.24).

In case $\operatorname{ch}\mathbb{K} \neq 2$ generic reducedness of Σ_W implies (generic) reducedness of Δ_W. The schemes Σ_W and Δ_W show similar behavior with respect to deletion and contraction (see Lemmas 4.28 and 4.30). As a consequence generic reducedness can be proved along the same lines (see Theorem 4.34). In both cases we have to show reducedness at all (the same) generic points. Our proof proceeds by induction over the cardinality of the matroid's underlying set and makes use of the handle decomposition.

In a first base case where the matroid is a single circuit, generic reducedness can be shown directly (see Lemma 4.32). In the other case the handle decomposition provides a *non-disconnective* handle H, which leaves the matroid connected when deleted (see Definition 2.2). In case $H = \{h\}$ is a handle of size 1, we show that Σ_W or Δ_W inherits reducedness from the corresponding scheme $\Sigma_{W\setminus h}$ or $\Delta_{W\setminus h}$ associated with the deletion $\mathsf{M}\setminus h$ (see Lemma 4.29).

The non-disconnective handle H provided by the handle decomposition is not unique. This leads us to consider non-disconnective handles independently of a handle decomposition. They turn out to be special instances of maximal handles which form the *handle partition* of the matroid (see Lemma 2.3). As a purely matroid-theoretic ingredient we show that the number of non-disconnective handles is strictly increasing when adding handles (see Proposition 2.8). This leads us to identify the prism matroid as a second base case (see Definition 2.18). Its handle partition consists of 3 non-disconnective handles of size 2 (see Lemmas 2.7 and 2.19). Here an explicit calculation shows that Δ_W is reduced in the torus $(\mathbb{K}^*)^6$ (see Lemma 4.27). The corresponding result for Σ_W holds if $\operatorname{ch}\mathbb{K} \neq 2$.

In the remaining case we use blowing-up, an ingredient from algebraic geometry. To this end we prove a result that recovers generic reducedness of a ring R *along* the subscheme defined by an ideal $I \lhd R$ (see Definition 4.3) from generic reducedness of the associated graded ring $\operatorname{gr}_I R$, the ring of the corresponding normal cone (see Lemma 4.5). We apply this result to the ring of Σ_W or Δ_W and a coordinate subscheme $V(x_F)$ defined by x_F for a partition $E = F \sqcup G$ (see Lemma 4.31). In this case the graded ring identifies with the ring of the respective scheme $\Sigma_{W/G}$ or $\Delta_{W/G}$ associated with the contraction M/G (see Lemma 4.30). Since we are assuming now that all non-disconnective handles H have size at least 2 there are at least 3 more edges than maximal handles (see Proposition 2.8). The case of equality is that of the prism matroid (see Lemmas 2.7 and 2.19). Using this inequality we construct a suitable partition $E = F \sqcup G$ for which *all* generic points of Σ_W or Δ_W are *along* $V(x_F)$ if the matroid is not the prism (see Lemma 4.33). This yields generic reducedness of Σ_W or Δ_W in this case. A slight modification of the approach finally covers the generic points outside the torus $(\mathbb{K}^*)^6$ in case of the prism matroid.

Finally consider 3-connected matroids M with $|E| > 3$. Here we prove that Σ_W is irreducible, which implies that Δ_W is integral (see Theorem 4.37). We first observe that handles of (co)size at least 2 yield

2-separations (see Lemma 2.3.(e)). It follows that the handle decomposition consists entirely of non-disconnective 1-handles (see Proposition 2.4) and that all generic points of Σ_W lie in $\mathbb{T}^E$ (see Corollary 4.26). We show that the number of generic points is bounded by that of $\Sigma_{W\setminus e}$ for all $e \in E$ (see Lemma 4.29). Duality switches deletion and contraction and identifies generic points of Σ_W and $\Sigma_{W^\perp}$ (see Corollary 4.15). Using Tutte's Wheels and Whirls Theorem this reduces irreducibility of Σ_W to the case where M is a wheel or whirl (see Lemma 4.38). We show that the n-wheel and n-whirl have the same configuration schemes X_W, Σ_W and Δ_W independent of W up to isomorphism (see Proposition 4.40). An induction on n with an explicit study of base cases finishes the proof (see Corollary 4.41 and Lemma 4.43).

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2. Matroids and realizations

Our algebraic objects of interest are associated to a realization of a matroid. In this section we prepare the path for an inductive approach driven by the underlying matroid structure. Our main tool is the handle decomposition, a matroid version of the ear decomposition of graphs.

2.1. **Matroid basics.** In the following we review the relevant basics of matroid theory using Oxley's book (see [Oxl11]) as a comprehensive reference.

Let M be a matroid on a set $E =: E_\mathsf{M}$. This consists of several collections of subsets of E which satisfy certain axioms, any one of which determine the others. In particular, these include the *independent sets*, denoted $\mathcal{I}_\mathsf{M} \subseteq 2^E$, the *bases*, $\mathcal{B}_\mathsf{M} \subseteq 2^E$, and the *circuits*, $\mathcal{C}_\mathsf{M} \subseteq 2^E$. By definition, the bases and circuits are respectively maximal independent and minimal dependent sets of $2^E \setminus \mathcal{I}_\mathsf{M}$ with respect to inclusion. By an *n-circuit* we mean a circuit with n elements, 3-circuits are called *triangles*.

The circuits define an equivalence relation on E where $e, f \in E$ are equivalent if $e, f \in C$ for some $C \in \mathcal{C}_\mathsf{M}$ (see [Oxl11, Prop. 4.1.2]). The corresponding equivalence classes are the *connected components* of M. If such a component is unique M is said to be *connected*.

An element $e \in E$ is a *loop* in M if $e \notin B$ for any $B \in \mathcal{B}_\mathsf{M}$, and a *coloop* if $e \in B$ for all $B \in \mathcal{B}_\mathsf{M}$. A matroid is *free* if every element is a coloop.

There is a *rank function* $\mathrm{rk}_{\mathsf{M}}\colon 2^E \to \mathbb{N}$ for which, in particular,

$$S \in \mathcal{I}_{\mathsf{M}} \iff \mathrm{rk}_{\mathsf{M}}(S) = |S|.$$

By definition, $\mathrm{rk}\,\mathsf{M} = \mathrm{rk}_{\mathsf{M}}(E)$.

The *connectivity function* $\lambda_{\mathsf{M}}\colon 2^E \to \mathbb{N}$ is defined by

$$\lambda_{\mathsf{M}}(S) := \mathrm{rk}(S) + \mathrm{rk}(E\backslash S) - \mathrm{rk}(\mathsf{M})$$

for any $S \subseteq E$. For $k > 0$ a subset $S \subseteq E$ is called a *k-separation* if

$$\lambda_{\mathsf{M}}(S) < k \leqslant \min\{|S|, |E\backslash S|\}.$$

The matroid M is said to be *k-connected* if it has no $(k-1)$-separations. In this case, a k-separation is called *exact*. Connectedness is the special case $k = 2$. It is not hard to show that k-connectedness is equivalent for M and $\mathsf{M}^\perp$ (see [Oxl11, Cor. 8.1.5]).

Here are some standard constructions of new matroids from old:

The *direct sum* $\mathsf{M}_1 \oplus \mathsf{M}_2$ of matroids M_1 and M_2 is the matroid on $E_{\mathsf{M}_1} \sqcup E_{\mathsf{M}_2}$ with independent sets

$$\mathcal{I}_{\mathsf{M}_1\oplus\mathsf{M}_2} := \{I_1 \sqcup I_2 \mid I_1 \in \mathcal{I}(\mathsf{M}_1), I_2 \in \mathcal{I}(\mathsf{M}_2)\}.$$

The sum is *proper* if $E_{\mathsf{M}_1} \neq \varnothing \neq E_{\mathsf{M}_2}$. Connectedness means that a matroid is not a proper direct sum (see [Oxl11, Cor. 4.2.9]).

For any subset $F \subseteq E$, the *restriction* matroid $\mathsf{M}|_F$ is the matroid on F defined by (see [Oxl11, 3.1.12])

$$\mathcal{I}_{\mathsf{M}|_F} := \{I \cap F \mid I \in \mathcal{I}_{\mathsf{M}}\}. \tag{2.1}$$

Its set of circuits is (see [Oxl11, 3.1.13])

$$\mathcal{C}_{\mathsf{M}|_F} = \mathcal{C}_{\mathsf{M}} \cap 2^F. \tag{2.2}$$

Thinking of restriction as an operation that deletes elements in F from E, one defines the *deletion* matroid $\mathsf{M}\backslash F := \mathsf{M}|_{E\backslash F}$. The *contraction* matroid M/F on $E\backslash F$ is defined by (see [Oxl11, Prop. 3.1.7])

$$\mathcal{I}_{\mathsf{M}/F} := \{I \subseteq E\backslash F \mid I \cup B \in \mathcal{I}_{\mathsf{M}}\}, \tag{2.3}$$

where B is any basis of $\mathsf{M}|_F$. Its circuits are the minimal non-empty sets $C\backslash F$ where $C \in \mathcal{C}_{\mathsf{M}}$ (see [Oxl11, Prop. 3.1.10]), that is,

$$\mathcal{C}_{\mathsf{M}/F} = \mathrm{Min}\,\{C\backslash F \mid F \not\supseteq C \in \mathcal{C}_{\mathsf{M}}\}. \tag{2.4}$$

Consider a bijection

$$\nu\colon E \to E^\vee, \quad e \mapsto e^\vee. \tag{2.5}$$

For any subset $S \subseteq E$, its complement in E can be identified with

$$S^\perp := \nu(E\backslash S) \subseteq E^\vee.$$

Then the *dual matroid* $\mathsf{M}^\perp$ is the matroid on $E^\vee$ whose bases are given by

$$\mathcal{B}_{\mathsf{M}^\perp} := \left\{B^\perp \;\middle|\; B \in \mathcal{B}_{\mathsf{M}}\right\}.$$

In particular $\operatorname{rk}\mathsf{M} + \operatorname{rk}\mathsf{M}^\perp = |E|$ (see [Oxl11, p. 2.1.8]). The k-connectivity of $\mathsf{M}^\perp$ coincides with that of M (see [Oxl11, Cor. 8.1.5]). For any subset $F \subset E$ (see [Oxl11, Ex. 3.1.1]), one can identify

$$(\mathsf{M}/F)^\perp = \mathsf{M}^\perp\backslash F, \quad (\mathsf{M}\backslash F)^\perp = \mathsf{M}^\perp/F. \tag{2.6}$$

Various matroid data of $\mathsf{M}^\perp$ is also considered as *co*data of M. A *triad* of M is a 3-cocircuit of M, that is, a triangle of $\mathsf{M}^\perp$.

The (codimension-1) *truncation* of M is, by definition, the matroid $T(\mathsf{M})$ on E with independent sets

$$\mathcal{I}_{T(\mathsf{M})} := \{S \in \mathcal{I}_\mathsf{M} \mid |S| \leqslant \operatorname{rk}\mathsf{M} - 1\}.$$

Example 2.1 (Uniform matroids and circuits). The *uniform matroid* of rank $r \geqslant 0$ on a set E of size $|E| = n$, denoted $\mathsf{U}_{r,n}$, has bases $\{B \subseteq E \mid |B| = r\}$. It has no loops or coloops if $0 < r < n$. By definition, $\mathsf{U}_{r,n}^\perp = \mathsf{U}_{n-r,n}$ for all $0 \leqslant r \leqslant n$.

Informally we refer to a matroid M on E for which $E \in \mathcal{C}_\mathsf{M}$ as a *circuit*, or as a *triangle* if $n = 3$. If $|E| = n$, then $\mathsf{U}_{n-1,n}$ is the unique such matroid. $\diamond$

2.2. **Handle decomposition.** In the following we investigate handles as building blocks of connected matroids.

Definition 2.2 (Handles)**.** Let M be a matroid. A subset $\varnothing \neq H \subseteq E$ is a *(proper) handle* in M if $C \cap H \neq \varnothing$ implies $H \subseteq C$ for all $C \in \mathcal{C}_\mathsf{M}$ (and $H \neq E$). By a *k-handle* we mean a handle of size k. It is *disconnective* if $\mathsf{M}\backslash H$ is disconnected. A subset $\varnothing \neq H' \subseteq H$ of a handle is called a *subhandle*. *Maximality* of handles refers to inclusion. Write $\mathcal{H}_\mathsf{M}$ for the set of handles in M, $\operatorname{Max}\mathcal{H}_\mathsf{M}$ for its subset of maximal handles. A handle $H \in \mathcal{H}_\mathsf{M}$ is called *separating* if $\min\{|H|, |E\backslash H|\} \geqslant 2$.

Singletons $\{e\}$ and subhandles are handles. If $\bigcup\mathcal{C}_\mathsf{M} \neq E$, then $E\backslash\bigcup\mathcal{C}_\mathsf{M} \in \operatorname{Max}\mathcal{H}_\mathsf{M}$ and is a union of coloops. The maximal handles in $\bigcup\mathcal{C}_\mathsf{M}$ are the minimal non-empty intersections of all subsets of $\mathcal{C}_\mathsf{M}$. Together they form the *handle partition* of E

$$E = \bigsqcup_{H \in \operatorname{Max}\mathcal{H}_\mathsf{M}} H,$$

which refines the partition of $\bigcup\mathcal{C}_\mathsf{M}$ into connected components.

For any subset $F \subseteq E$, $\mathcal{H}_\mathsf{M} \cap 2^F \subseteq \mathcal{H}_{\mathsf{M}|_F}$ by (2.2).

Lemma 2.3. *Let M be a matroid and $H \in \mathcal{H}_\mathsf{M}$.*

(a) If $H = E$, then $\mathsf{M} = \mathsf{U}_{r,n}$ where $n = |E|$ and $r \in \{n-1, n\}$ (see Example 2.1). In the latter case $|E| = 1$ or M is disconnected.

(b) Either $H \in \mathcal{I}_\mathsf{M}$ or $H \in \mathcal{C}_\mathsf{M}$. In the latter case H is a connected component of M. In particular, if M is connected and H is proper, then $H \in \mathcal{I}_\mathsf{M}$ and $H \subsetneq C$ for some circuit $C \in \mathcal{C}_\mathsf{M}$.

(c) For any $\varnothing \neq H' \subseteq H$, $H\backslash H'$ consists of coloops in $\mathsf{M}\backslash H'$. In particular, non-disconnective handles are maximal.

(d) *If $H \notin \mathcal{C}_{\mathsf{M}}$, then $\mathcal{C}_{\mathsf{M}} \to \mathcal{C}_{\mathsf{M}/H}$, $C \mapsto C\backslash H$, is a bijection. If $H \notin \operatorname{Max}\mathcal{H}_{\mathsf{M}}$, then $\operatorname{Max}\mathcal{H}_{\mathsf{M}} \to \operatorname{Max}\mathcal{H}_{\mathsf{M}/H}$, $H' \mapsto H'\backslash H$, is a bijection which identifies non-disconnective handles. In this case, the connected components of M which are not contained in $H\backslash\bigcup\mathcal{C}_{\mathsf{M}}$ correspond to the connected components of M/H.*

(e) *Suppose M is connected and H is proper. Then $\operatorname{rk}(\mathsf{M}/H) = \operatorname{rk}\mathsf{M} - |H|$ and $\lambda_{\mathsf{M}}(H) = 1$. In particular, if H is separating, then $H \sqcup (E\backslash H)$ is a 2-separation of M.*

Proof.
(a) Suppose $H = E$. Then $\mathcal{C}_{\mathsf{M}} \subseteq \{E\}$ and $M = \mathsf{U}_{n-1,n}$ in case of equality. Otherwise $\mathcal{C}_{\mathsf{M}} = \varnothing$ implies $\mathcal{B}_{\mathsf{M}} = \{E\}$ and $M = \mathsf{U}_{n,n}$ (see [Oxl11, Prop. 1.1.6]).

(b) Suppose $H \notin \mathcal{I}_{\mathsf{M}}$. Then there is a circuit $H \supseteq C \in \mathcal{C}_{\mathsf{M}}$. By definition of handle and incomparability of circuits, $H = C$ is disjoint from all other circuits and hence a connected component of M.

(c) Let $d \in H\backslash H'$. If d is not a coloop in $\mathsf{M}\backslash H'$, then $d \in C \cap H$ for some $C \in \mathcal{C}_{\mathsf{M}\backslash H'} \subseteq \mathcal{C}_{\mathsf{M}}$ (see (2.2)). Hence $H' \subseteq H \subseteq C$ since H is a handle, a contradiction.

(d) The first bijection follows from (2.4) with $F = H$. The remaining claims follow from the discussion preceding the lemma.

(e) Part (b) yields the first equality (see [Oxl11, Prop. 3.1.6]) along with a circuit $H \neq C \in \mathcal{C}_{\mathsf{M}}$. Now let B be a basis of $\mathsf{M}\backslash H$, and let $S = B \cup H$. Clearly S spans M. For any $e \in H$, we check $S\backslash\{e\}$ is independent: if not, $S\backslash\{e\}$ contains a circuit C. Since $C \nsubseteq B$, we have $H \cap C \neq \varnothing$ and hence $e \in H \subseteq C$, a contradiction. It follows that $\operatorname{rk}\mathsf{M} = |S| - 1 = \operatorname{rk}(\mathsf{M}\backslash H) + |H| - 1$ and hence the second equality. ☐

Proposition 2.4. *(Handles in 3-connected matroids) Let M be a 3-connected matroid on E with $|E| > 3$. Then all its handles are non-disconnective 1-handles.*

Proof. Let $H \in \mathcal{H}_{\mathsf{M}}$ be any handle. By Lemma 2.3.(a), H must be proper. Note that M cannot be a circuit and hence $|E\backslash H| \geqslant 2$ by Lemma 2.3.(b). Then H is a 1-handle as otherwise H yields a 2-separation of M by Lemma 2.3.(e).

Suppose that H is disconnective. Consider the deletion $\mathsf{M}' := \mathsf{M}\backslash H$ on the set $E' := E\backslash H$. Pick a minimal connected component X of M'. Since $H \neq \varnothing$ and $|E| > 3$ both $X \cup H$ and its complement $E\backslash(X \cup H) = E'\backslash X$ have at least 2 elements.

Since X is a connected component of M' and by Lemma 2.3.(e),

$$\operatorname{rk}(X) + \operatorname{rk}(E'\backslash X) = \operatorname{rk}\mathsf{M}' = \operatorname{rk}\mathsf{M}.$$

Since $\operatorname{rk}(X \cup H) \leqslant \operatorname{rk}(X) + |H| = \operatorname{rk} X + 1$ it follows that

$$\operatorname{rk}(X \cup H) + \operatorname{rk}(E\backslash(X \cup H)) \leqslant \operatorname{rk}\mathsf{M} + 1.$$

Whence $X \cup H$ is a 2-separation, a contradiction. ☐

The following result is the basis for our inductive approach to connected matroids.

Proposition 2.5. *(Handle decomposition) Let* M *be a connected matroid and* $C_1 \in \mathcal{C}_{\mathsf{M}}$*. Then there is a filtration* $C_1 = F_1 \subsetneq \cdots \subsetneq F_k = E$ *such that* $\mathsf{M}|_{F_i}$ *is connected and* $H_i := F_i \backslash F_{i-1} \in \mathcal{H}_{\mathsf{M}|_{F_i}}$*, for* $i = 2, \ldots, k$*.*

Proof. A *handle (or ear) decomposition* of a matroid M is a collection of circuits $C_1, \ldots, C_k$ such that, for $F_i = \bigcup_{j \leqslant i} C_j$ we have $C_i \cap F_{i-1} \neq \varnothing$, and $C_i \backslash F_{i-1}$ is a circuit in M/F_{i-1} for $i = 2, \ldots, k$. If M is connected, then M has a handle decomposition with arbitrary C_1 (see [CH96]), and the hypothesis $C_i \cap F_{i-1} \neq \varnothing$ implies that $\mathsf{M}|_{F_i}$ is connected for each $i = 1, \ldots, k$.

It remains to check that H_i is a handle in $\mathsf{M}|_{F_i}$ for $i = 2, \ldots, k$. Since circuits are nonempty, $\varnothing \neq H_i \subsetneq F_i$. Now choose any $e \in H_i = C_i \backslash F_{i-1}$. If C is a circuit containing e, suppose by way of contradiction that $C \not\supseteq H_i$. Then there exists some $d \in C_i \backslash (C \cup F_{i-1})$. By the strong circuit exchange axiom (see [Oxl11, §1.1, Ex. 14]), there is another circuit C' contained in F_i for which $d \in C' \subseteq (C \cup C_i) \backslash \{e\}$. But then $C' \backslash F_{i-1} \subseteq C_i \backslash F_{i-1}$ because $C' \subseteq F_i$. Since C_i is assumed to be a circuit of M/F_{i-1}, it follows that either $C' \subseteq F_{i-1}$ or $C' \backslash F_{i-1} = C_i \backslash F_{i-1}$ (see (2.4)). The former is impossible because $C' \ni d \notin F_{i-1}$, and the latter is impossible because $C' \cup F_{i-1} \not\ni e \in C_i$. ◻

In the sequel we develop a bound for the number of non-disconnective handles.

Lemma 2.6. *Let* M *be a connected matroid.*

(a) If $H \in \mathcal{H}_{\mathsf{M}}$ *and* $H' \in \mathcal{H}_{\mathsf{M} \backslash H}$ *are non-disconnective with* $H \cup H' \neq E$*, then there is a non-disconnective handle* $H'' \in \mathcal{H}_{\mathsf{M}}$ *for which* $H'' \subseteq H'$*, with equality if* $H' \in \mathcal{H}_{\mathsf{M}}$*.*

(b) If $H, H' \in \mathcal{H}_{\mathsf{M}}$ *with* $E \neq H \cup H' \in \mathcal{C}_{\mathsf{M}}$*, then* H *and* H' *are not disconnective.*

Proof.

(a) By hypothesis, M and $\mathsf{M} \backslash H$ are connected and $H \cup H' \neq E$. Then, using that H and H' are handles, there are circuits $C \in \mathcal{C}_{\mathsf{M}}$ and $C' \in \mathcal{C}_{\mathsf{M} \backslash H}$ with $H \subsetneq C$ and $H' \subsetneq C'$.

Suppose that $C \subseteq H \cup H'$. Then the strong circuit exchange axiom (see [Oxl11, §1.1, Ex. 14]) yields a circuit $C'' \in \mathcal{C}_{\mathsf{M}}$ for which $C'' \subseteq H \cup C'$, $H' \nsubseteq C''$ and $C'' \nsubseteq H \cup H'$. Since $C'' \subsetneq C'$ contradicts incomparability of circuits, $H \subsetneq C''$ since H is a handle and Lemma 2.3.(b) forbids equality.

Replacing C by C'' if necessary, then, we may assume that $C \nsubseteq H \cup H'$. By hypothesis, $\mathsf{M} \backslash (H \cup H')$ is connected, and C witnesses the fact that H, $C \cap H'$ and $E \backslash (H \cup H')$ are all in the same connected component. Then the set $H'' := H' \backslash C$ is in $\mathcal{H}_{\mathsf{M} \backslash H}$, and $\mathsf{M} \backslash H''$ is connected.

If $H'' \notin \mathcal{H}_{\mathsf{M}}$ there is a circuit $C'' \in \mathcal{C}_{\mathsf{M}}$ such that $\varnothing \neq C'' \cap H'' \neq H''$. In particular $H \subseteq C''$, since otherwise C'' is disjoint from H, and $C'' \in \mathcal{C}_{\mathsf{M}} \cap 2^{E\backslash H} = \mathcal{C}_{\mathsf{M}\backslash H}$, which would contradict $H' \in \mathcal{H}_{\mathsf{M}\backslash H}$. This means that C'' connects H with $C'' \cap H''$. We may therefore replace H'' by $H''\backslash C'' \subsetneq H''$ and iterate. After finitely many steps, then, $H'' \in \mathcal{H}_{\mathsf{M}}$.

(b) Set $C := H \cup H'$ and let $d \in E\backslash C$ and $e \in H$. By connectedness of M, there is a $C' \in \mathcal{C}_{\mathsf{M}}$ such that $d, e \in C'$. Then $e \in C' \cap H$ and hence $H \subseteq C'$ since H is a handle. Assume that $C' \cap H' \neq \varnothing$. Then also $H' \subseteq C'$ since H' is a handle. Thus $d \notin C = H \cup H' \subsetneq C' \ni d$ contradicting incomparability of circuits. Therefore $C' \cap H' = \varnothing$ and hence $d, e \in C' \in \mathcal{C}_{\mathsf{M}} \cap 2^{E\backslash H'} = \mathcal{C}_{\mathsf{M}\backslash H'}$. It follows that $\mathsf{M}\backslash H'$ is connected. ☐

Lemma 2.7. *Let M be a connected matroid with a handle decomposition of length 2. Then M has at least 3 (disjoint) non-disconnective handles. In case of equality they form the handle partition.*

Proof. With notation from (the proof of) Proposition 2.5 consider the circuits $C' := C_1 \in \mathcal{C}_{\mathsf{M}}$, $C := C_2 \in \mathcal{C}_{\mathsf{M}}$, the handle $H := H_2 \in \mathcal{H}_{\mathsf{M}}$ and the subsets $\varnothing \neq H' := C'\backslash C \subseteq E$ and $\varnothing \neq H'' := C \cap C' \subseteq E$. Then $E = H \sqcup H' \sqcup H''$ and $C' = H' \cup H''$ and $C = H \cup H''$.

Let $C'' \in \mathcal{C}_{\mathsf{M}}$ be a circuit with $C' \neq C'' \neq C$. By Lemma 2.3.(d), we may assume that $|H| = 1$. Then $H' \subseteq C''$ (see [Oxl11, §1.1, Ex. 5]) and hence $H' \in \mathcal{H}_{\mathsf{M}}$. In case $H'' \in \mathcal{H}_{\mathsf{M}}$ is a handle, the proof is complete.

By incomparability of circuits, $C'' \nsubseteq C'$ and hence $H \subseteq C''$ since H is a handle. Thus, $H \cup H' \subseteq C''$ for any circuit $C'' \in \mathcal{C}_{\mathsf{M}}$ be a circuit with $C' \neq C'' \neq C$.

Suppose that $H' \notin \mathcal{H}_{\mathsf{M}}$ and pick C'' such that $\varnothing \neq C'' \cap H'' \neq H''$. Then C'' connects $C'' \cap H''$ with H' and H. Again the proof is complete if $H''\backslash C'' \in \mathcal{H}_{\mathsf{M}}$ is a handle. Otherwise iterating yields a handle $H''\backslash C'' \supseteq H''' \in \mathcal{H}_{\mathsf{M}}$. By the strong circuit exchange axiom (see [Oxl11, §1.1, Ex. 14]), there is a different C'' such that $H''' \subseteq C''$. Then repeating the preceding argument yields a fourth non-disconnective handle $H''\backslash H''' \supseteq H'''' \in \mathcal{H}_{\mathsf{M}}$. ☐

Proposition 2.8. *(Number of non-disconnective handles) Let M be a connected matroid with a handle decomposition of length $k \geqslant 2$ as in Proposition 2.5. Then M has at least $k+1$ (disjoint) non-disconnective handles.*

Proof. We argue by induction, the base case $k = 2$ being covered by Lemma 2.7. Let $k > 2$ and assume the claim holds for matroids with handle decompositions of length up to $k-1$. Suppose M is connected and has a handle decomposition of length k with non-disconnective handle $H_k = E - F_{k-1} \in \mathcal{H}_{\mathsf{M}}$. By induction, $\mathsf{M}\backslash H = \mathsf{M}|_{F_{k-1}}$ has at least k non-disconnective handles $H'_0, \dots, H'_{k-1} \in \mathcal{H}_{\mathsf{M}\backslash H}$. By Lemma 2.3.(a) and (c), $H'_i \neq E\backslash H$ and hence $H'_i \in \operatorname{Max} \mathcal{H}_{\mathsf{M}\backslash H}$ for $i = 1, \dots, k-1$.

In particular the $H'_0, \dots, H'_{k-1}$ are disjoint and also $H_k \cup H'_i \neq E$ for $i = 1, \dots, k-1$. Lemma 2.6.(a) now yields for each $i = 1, \dots, k-1$ a non-disconnective handle $H'_i \supseteq H''_i \in \mathcal{H}_{\mathsf{M}}$. Finally M has $k+1$ non-disconnective handles $H''_0, \dots, H''_{k-1}, H_k$. $\square$

We conclude this section with an observation.

Lemma 2.9. *Let M be a connected matroid of rank $\operatorname{rk} \mathsf{M} \geqslant 2$. Then there is a circuit $C \in \mathcal{C}_{\mathsf{M}}$ of size $|C| \geqslant 3$.*

Proof. Suppose instead all circuits have at most 2 elements. Since a circuit of size k has rank $k-1$, the union of all circuits containing any element e would equal the closure of e ([Oxl11, Prop. 1.4.11.(ii)]). So the closure of e would be a connected component of M ([Oxl11, Prop. 4.1.2]), hence all of E, by our assumption that M is connected. But then M has rank 1, a contradiction. $\square$

2.3. **Configurations and realizations.** Our objects of interest are not associated to a matroid itself but a realization as defined in the following. All matroid operations come with a counter-part for realizations.

Fix a field $\mathbb{K}$ and denote the $\mathbb{K}$-dual by $-^\vee := \operatorname{Hom}_{\mathbb{K}}(-, \mathbb{K})$. For a set E consider $\mathbb{K}^E$ as a *based* $\mathbb{K}$*-vector space* with basis E. Denote by $E^\vee = (e^\vee)_{e \in E}$ the dual basis.

We define configurations following Bloch, Esnault and Kreimer (see [BEK06, §1]).

Definition 2.10 (Configurations)**.** Let E be a set. A $\mathbb{K}$-vector subspace $W \subseteq \mathbb{K}^E$ is called a *configuration* (over $\mathbb{K}$). It is called *totally unimodular* if it admits a basis with all determinants of the coefficient matrix 0 or ± 1. It defines a matroid M_W on E with independent sets

$$\mathcal{I}_{\mathsf{M}_W} = \{S \subseteq E \mid (e^\vee|_W)_{e \in S} \text{ is } \mathbb{K}\text{-linearly independent in } W^\vee\}.$$

Remark 2.11 (Hyperplane arrangements)*.* A configuration in the sense of Definition 2.10 is in fact a configuration of vectors $e^\vee|_W \in W^\vee$, for $e \in E$. Suppose that $e^\vee \neq 0$ for each $e \in E$ or, equivalently, that M_W has no loops. Then the images of the $e^\vee|_W$ in $\mathbb{P}W^\vee$ form a projective point configuration in the classical sense (see [HC52]). Dually, the hyperplanes $\ker(e^\vee) \cap W$ form a *hyperplane arrangement* in W (see [OT92]), which is an equivalent notion in this case. $\diamond$

Definition 2.12 (Realizations)**.** Let M be a matroid and $W \subseteq \mathbb{K}^E$ a configuration (over $\mathbb{K}$). If $\mathsf{M} = \mathsf{M}_W$, then W is called a *(linear) realization* of M and M is called *(linearly) realizable* (over $\mathbb{K}$). If M admits a realization over $\mathbb{K} = \mathbb{F}_2$, then it is called a *binary* matroid. If M admits a totally unimodular realization, then it is called a *regular* matroid.

Remark 2.13 (Matroids and linear algebra). Given a realization $W \subseteq \mathbb{K}^E$ of M, the notions in §2.1 are derived from linear (in)dependence over $\mathbb{K}$. For example, for any subset $S \subseteq E$ and defining matrix A of W, the rank $\operatorname{rk}_{\mathsf{M}}(S)$ equals the rank of the submatrix of A with columns S. In particular, taking $S = E$, we note that $\operatorname{rk} \mathsf{M} = \dim W$. An element $e \in E$ is a loop if and only if column e of A is zero; e is a coloop if and only if column e is not in the span of the other columns. ◇

We fix some notation for realizations of basic matroid operations. Any subset $S \subseteq E$ gives rise to an inclusion and a projection

$$\iota_S \colon \mathbb{K}^S \hookrightarrow \mathbb{K}^E, \quad \pi_S \colon \mathbb{K}^E \twoheadrightarrow \mathbb{K}^E/\mathbb{K}^{E\setminus S} = \mathbb{K}^S$$

of based $\mathbb{K}$-vector spaces.

Definition 2.14 (Realizations of matroid operations)**.** Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M.

(a) The *dual* matroid $\mathsf{M}^\perp$ is realized by the configuration

$$W^\perp := (\mathbb{K}^E/W)^\vee \subseteq (\mathbb{K}^E)^\vee = \mathbb{K}^{E^\vee}.$$

(b) For $0 \neq \varphi \in W^\vee$ consider the *hyperplane configuration*

$$W_\varphi := \ker \varphi \subseteq \mathbb{K}^E.$$

(c) The configuration

$$\begin{aligned} W|_F &:= \pi_F(W) \subseteq \mathbb{K}^F \\ &\cong (W + \mathbb{K}^{E\setminus F})/\mathbb{K}^{E\setminus F} \cong W/(W \cap \mathbb{K}^{E\setminus F}) \end{aligned}$$

realizes the *restriction* matroid $\mathsf{M}|_F$.

(d) The configuration

$$W\setminus F := W|_{E\setminus F}$$

realizes the *deletion* matroid $\mathsf{M}\setminus F$. We abbreviate $W\setminus e := W\setminus\{e\}$.

(e) The configuration

$$W/F := W \cap \mathbb{K}^{E\setminus F} \subseteq \mathbb{K}^{E\setminus F}$$

realizes the *contraction* matroid M/F.

Remark 2.15. Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M.

(a) The element $e \in E$ is a loop or coloop of M if and only if $W \subseteq \mathbb{K}^{E\setminus\{e\}}$ or $W = (W\setminus e) \oplus \mathbb{K}^{\{e\}}$ respectively. In these cases $W\setminus e = W/e \subseteq \mathbb{K}^{E\setminus\{e\}}$.

(b) If $\pi_B(\ker \varphi \cap W) \neq \mathbb{K}^B$ for each $B \in \mathcal{B}_{\mathsf{M}}$, then $\mathsf{M}_{W_\varphi} = T(\mathsf{M})$. ◇

Example 2.16 (Realizations of uniform matroids). If W is the row span of a $r \times n$ matrix which is generic in the sense that all its maximal minors are non-zero, then W is a realization of the uniform matroid $\mathsf{U}_{r,n}$ (see Example 2.1). ◇

2.4. **Graphic matroids.** Matroids arising from graphs are the most prominent examples for our results.

A *graph* $G = (V, E)$ is a pair of finite sets V and E of vertices and edges where each edge $e \in E$ is a set of one or two vertices in V. This allows for multiple edges between pairs of vertices, and loops at vertices. For simplicity we consider only *connected* graphs.

A graph determines a *graphic matroid* $\mathsf{M}(G)$ on E by declaring a subset $S \subseteq E$ to be an independent set if the edge-induced subgraph S is acyclic. The bases of $\mathsf{M}(G)$ are the *spanning trees* of G (see [Oxl11, p. 18]),

$$\mathcal{B}_{\mathsf{M}_G} = \mathcal{T}(G). \tag{2.7}$$

Recall that a vertex in a connected graph is a cut vertex if its removal disconnects the graph. We remark that the matroid $\mathsf{M}(G)$ of a connected graph G with at least three vertices is connected if and only if G has no cut vertex (see [Oxl11, Cor. 8.1.6]). We refer also to [Oxl11, Ch. 8] for a complete discussion of notions of graph connectivity versus matroid connectivity.

Graphic matroids have linear realizations coming from their edge-vertex incidence matrices, as follows (see [BEK06, §2]). A choice of orientation turns G into a CW-complex. This gives rise to an exact sequence

(2.8)
$$\begin{array}{c} 0 \longrightarrow H_1(G, \mathbb{K}) \longrightarrow \mathbb{K}^E \xrightarrow{\ \delta\ } \mathbb{K}^V \xrightarrow{\ \sigma\ } H_0(G, \mathbb{K}) \longrightarrow 0 \\ (s \to t) \longmapsto t - s \end{array}$$

with dual

$$0 \longleftarrow H^1(G, \mathbb{K}) \longleftarrow \mathbb{K}^E \xleftarrow{\ \delta^\vee\ } \mathbb{K}^V \longleftarrow H^0(G, \mathbb{K}) \longleftarrow 0$$

Definition 2.17 (Graph configuration)**.** We call $W_G := \operatorname{Im} \delta^\vee$ the *graph configuration* of the graph G over $\mathbb{K}$.

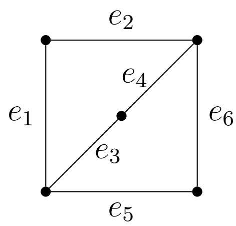
The subspace $W_G \subseteq \mathbb{K}^E$ is a totally unimodular realization of $\mathsf{M}(G)$ (see [Oxl11, Lem. 5.1.3]) and independent of the chosen orientation on G. By construction, $W_G^\perp = H_1(G, \mathbb{K})$ realizes its dual $\mathsf{M}(G)^\perp$ (see Definition 2.14.(a)).

Besides circuits (see Example 2.1) the following matroid is a base case of our inductive approach.

Definition 2.18 (Prism matroid)**.** We call the matroid associated with the $(2, 2, 2)$-theta graph (see Figure 1) the *prism matroid*, since it can also be realized as the six vertices of a triangle-based prism in $\mathbb{P}^3$.

Lemma 2.19 (Characterization of the prism matroid)**.** *Let* M *be a connected matroid on* $E = \{e_1, \dots, e_6\}$ *with* $|E| = 6$ *whose handle partition* $E = H_1 \sqcup H_2 \sqcup H_3$ *is made of* 3 *maximal* 2*-handles* $H_1 = \{e_1, e_2\}$*,* $H_2 = \{e_3, e_4\}$ *and* $H_3 = \{e_5, e_6\}$ *(see Lemma 2.7). Then* M *is*

FIGURE 1. Graph defining the prism matroid.



the prism matroid. Up to scaling E, it has a unique realization W with basis

$$w^1 := e_1 + e_2, \quad w^2 := e_3 + e_4, \quad w^3 := e_5 + e_6, \quad w^4 := e_1 + e_3 + e_5.$$

Proof. Each circuit is a union of handles. By Lemma 2.3.(b), no H_i is a circuit but each H_i is properly contained in one. After renumbering this yields circuits $C_1 = H_2 \sqcup H_3$ and $C_2 = H_1 \sqcup H_3$. The strong circuit exchange axiom (see [Oxl11, §1.1, Ex. 14]) yields a third circuit $C_3 = H_1 \sqcup H_2$. However, if E is a circuit, then it is unique and E is the unique maximal handle. Therefore $\mathcal{C}_{\mathsf{M}} = \{C_1, C_2, C_3\}$ coincides with the circuits of the prism matroid. The first claim follows.

Let W be any realization of M. By the above, $\dim W = \operatorname{rk} \mathsf{M} = 4$. Pick a basis $w^i = \sum_{j=1}^6 w^i_j e_j$, $i = 1, \ldots, 4$. We may assume that columns $2, 4, 6, 5$ of the coefficient matrix $(w^i_j)_{i,j}$ form an identity matrix. Since C_1 and C_2 are circuits, $w^1_3 = 0 \neq w^2_3$ and $w^2_1 = 0 \neq w^1_1$. Thus,

$$(w^i_j)_{i,j} = \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 1 & 0 & 0 \\ * & 0 & * & 0 & 0 & 1 \\ * & 0 & * & 0 & 1 & 0 \end{pmatrix}.$$

Since C_3 is a circuit, suitably replacing $w^3, w^4 \in \langle w^3, w^4 \rangle$, reordering H_3 and scaling e_1, e_3 makes

$$(w^i_j)_{i,j} = \begin{pmatrix} * & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

where $w^1_1, w^2_3, w^3_5 \neq 0$. Now suitably scaling first w^1, w^2, w^3 and then e_2, e_4, e_6 makes

$$(w^i_j)_{i,j} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

The second claim follows. $\square$

The following classes of matroids play a distinguished role in connection with 3-connectedness.

Example 2.20 (Wheels and whirls). We recall from [Oxl11, §8.4] the *wheel* and *whirl matroids*. For $n \geqslant 2$ the *wheel graph* G_n is obtained from an n-cycle, the “rim”, by adding an additional vertex and edges, the “spokes”, joining it to each vertex in the rim (see Figure 2). We write S for the set of spokes and R for the set of edges in the rim.

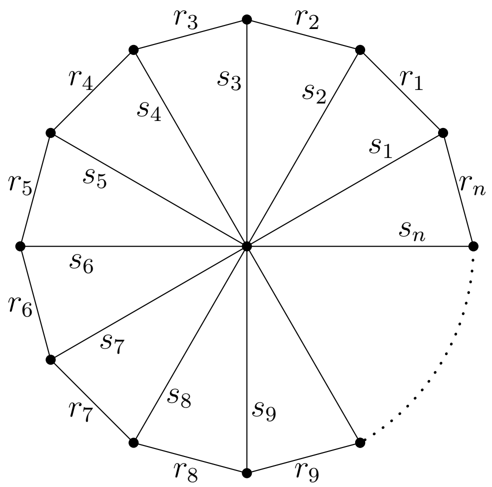


FIGURE 2. The wheel graph G_n.

For $n \geqslant 3$ the *wheel matroid* is the graphic matroid $\mathsf{W}_n := \mathsf{M}(G_n)$ on $E := S \sqcup R$. For $n \geqslant 2$ the *whirl matroid* is the (non-graphic) matroid on E obtained from $\mathsf{M}(G_n)$ by *relaxation* of the rim, that is,

$$\mathcal{B}_{\mathsf{W}^n} = \mathcal{B}_{\mathsf{M}(G_n)} \sqcup \{R\}.$$

In terms of circuits this means that

$$\mathcal{C}_{\mathsf{W}^n} = \mathcal{B}_{\mathsf{M}(G_n)} \backslash R \sqcup \{\{s\} \sqcup R \mid s \in S\}.$$

We use a cyclic index set $\{1, \ldots, n\} = \mathbb{Z}_n$ and write $S = \{s_1, \ldots, s_n\}$ and $R = \{r_1, \ldots, r_n\}$. Then $\{s_i, r_i, s_{i+1}\}$ and $\{r_i, r_{i+1}, s_{i+1}\}$ are triangles and triads respectively. In fact, this property enforces $\mathsf{M} \in \{\mathsf{W}_n, \mathsf{W}^n\}$ for any connected matroid M on $E \sqcup F$ (see [Sey80, (6.1)]).

In Lemma 4.39 we describe all realizations of wheels and whirls. In particular it shows the well-known fact that whirls are not binary. $\diamond$

3. Configuration polynomials and forms

In this section we define configuration polynomials and configuration forms. We lay the foundation for an inductive proof of our main result using a handle decomposition. In the process we generalize some known results on graph polynomials to configuration polynomials.

3.1. **Configuration polynomials.** To prepare the definition of configuration polynomials we introduce some notation.

Let $W \subseteq \mathbb{K}^E$ be a configuration. Compose the associated inclusion map with π_S to a map

$$\alpha_{W,S}\colon W \hookrightarrow \mathbb{K}^E \xrightarrow{\pi_S} \mathbb{K}^S \tag{3.1}$$

invariant under enlarging E. Fix an isomorphism

$$c_W\colon \mathbb{K} \xrightarrow[\cong]{} \bigwedge^{\dim W} W \tag{3.2}$$

and set $c_0 := \mathrm{id}_{\mathbb{K}}$. Note that a choice of basis of W gives rise to such an isomorphism. Fix an ordering on E to identify

$$\bigwedge^{|S|} \mathbb{K}^S = \mathbb{K}. \tag{3.3}$$

Note that different orderings result in a sign change only. If S has size $|S| = \dim W$, consider the determinant

$$\det \alpha_{W,S}\colon \mathbb{K} \xrightarrow[\cong]{c_W} \bigwedge^{|S|} W \xrightarrow{\bigwedge^{|S|} \alpha_{W,S}} \bigwedge^{|S|} \mathbb{K}^S = \mathbb{K}$$

defined up to sign and set

$$c_{W,S} := \det{}^2 \alpha_{W,S} \in \mathbb{K}.$$

Note that $\alpha_{0,\varnothing} = \mathrm{id}_{\mathbb{K}}$ and hence $c_{0,\varnothing} = 1$.

Remark 3.1. Let $W \subseteq \mathbb{K}^E$ be a configuration, and let $S \subseteq F \subseteq E$ with $|S| = \dim W$. Then the maps (3.1) for W and $W|_F$ form a commutative diagram

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[rr, bend left, "\alpha_{W,S}"] \arrow[d, "\pi_F"', "\cong"] & \mathbb{K}^E \arrow[r, "\pi_S"] \arrow[d, "\pi_F"'] & \mathbb{K}^S \arrow[d, equal] \\
W|_F \arrow[r, hook] \arrow[rr, bend right, "\alpha_{W|_F,S}"'] & \mathbb{K}^E \arrow[r, "\pi_S"] & \mathbb{K}^S
\end{tikzcd}$$

and hence $c_{W,S} = c^2 \cdot c_{W|_F,S}$ for some $c \in \mathbb{K}^*$ independent of S. ◇

Consider the dual basis $E^\vee = \{e^\vee \mid e \in E\}$ of E as coordinates

$$x_e := e^\vee, \quad e \in E, \tag{3.4}$$

on $\mathbb{K}^E$, and abbreviate $\partial_e := \frac{\partial}{\partial x_e}$. Given an enumeration of $E = \{e_1, \dots, e_n\}$ we write $x_i := x_{e_i}$ and $\partial_i := \partial_{e_i}$. For a subset $S \subseteq E$, set $x_S := (x_e)_{e \in S}$ and $x^S := \prod_{e \in S} x_e$ and abbreviate $x := x_E$.

Definition 3.2 (Configuration polynomials)**.** The *configuration polynomial* of the configuration $W \subseteq \mathbb{K}^E$ is the polynomial

$$\psi_W := \sum_{B \in \mathcal{B}_{\mathsf{M}}} c_{W,B} \cdot x^B \in \mathbb{K}[x].$$

Remark 3.3 (Well-definedness of configuration polynomials). Any two isomorphisms (3.2) differ by a nonzero multiple $c \in \mathbb{K}^*$. Using the isomorphism $c \cdot c_W$ in place of c_W replaces ψ_W by $c^2 \cdot \psi_W$. In other words, ψ_W is well-defined up to a squared non-zero factor. Whenever ψ_W occurs in a formula, we mean that the formula holds true for a suitable choice of such a factor. ◇

Remark 3.4 (Configuration polynomials and basis scaling). Dividing $e \in E$ by $c \in \mathbb{K}^*$ multiplies $x_e = e^\vee$ by c (see Remark 2.11) and the identifications (3.3) with $e \in S$ by c. This results in multiplying, for each $e \in B \in \mathcal{B}_{\mathsf{M}}$, $c_{W,B}$ by c^2 and x^B by c. The same result is achieved by substituting $c^3 \cdot x_e$ for x_e in ψ_W. Scaling E thus results in scaling x in ψ_W.

On the other hand, dropping the equality (3.4) and scaling $e \in E$ for fixed x_e replaces W in ψ_W by an equivalent realization (see [Oxl11, §6.3]). ◇

Remark 3.5 (Degree of configuration polynomials). By definition, $\operatorname{rk} \mathsf{M} = 0$ if and only if $\psi_W = 1$ for some/any realization $W \subseteq \mathbb{K}^E$ of M and otherwise

$$\deg \psi_W = \operatorname{rk} \mathsf{M} = \dim W$$

for any realization $W \subseteq \mathbb{K}^E$ of M. A variable x_e does not appear in (divides) ψ_W exactly if $e \in E$ is a (co)loop in M. ◇

Remark 3.6 (Matroid polynomial). For any matroid M, not necessarily realizable, one might consider the *matroid (basis) polynomial*

$$\psi_{\mathsf{M}} := \sum_{B \in \mathcal{B}_{\mathsf{M}}} x^B.$$

If M is regular, then $\psi_W = \psi_{\mathsf{M}}$ for any totally unimodular realization W of M. In this case, for any field $\mathbb{K}$, all realizations of M are equivalent (see [Oxl11, Prop. 6.6.5]), and thus define geometrically equivalent configuration polynomials (see Remark 3.4). In general, ψ_W and ψ_{M} are geometrically different (see Example 5.2). ◇

Example 3.7 (Configuration polynomials of free matroids and circuits). Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M, and set $n := |E|$.

(a) Suppose that M is free. Then then $E \in \mathcal{B}_{\mathsf{M}}$ and

$$\psi_W = x^E$$

is the elementary symmetric polynomial of degree n in n variables.

(b) Suppose that M is a circuit. Then $E \in \mathcal{C}_{\mathsf{M}}$ and by Remark 3.1

$$\psi_W = \sum_{e\in E} \psi_{W\setminus e}.$$

With $E = \{e_1, \dots, e_n\}$, W has a basis $w^i = e_i + c_i \cdot e_n$ with $c_i \in \mathbb{K}^*$ where $i = 1, \dots, n-1$. Scaling first $w^1, \dots, w^{n-1}$ and then $e_1, \dots, e_{n-1}$ makes $c_1 = \cdots = c_{n-1} = 1$. This makes ψ_W the elementary symmetric polynomial of degree $n-1$ in n variables. $\diamond$

Example 3.8 (Configuration polynomial of the prism). For the unique realization W of the prism matroid (see Lemma 2.19),

$$\psi_W = x_1x_2(x_3+x_4)(x_5+x_6)+x_3x_4(x_1+x_2)(x_5+x_6)+x_5x_6(x_1+x_2)(x_3+x_4).$$

$\diamond$

In the following we put matroid connectivity in correspondence with irreducibility of configuration polynomials. As a preparation we lift a direct sum of matroids to any realization.

Lemma 3.9. *Any decomposition* $\mathsf{M} = \mathsf{M}_1 \oplus \mathsf{M}_2$ *of matroids with underlying partition* $E = E_1 \sqcup E_2$ *induces a decomposition of realizations* $W = W_1 \oplus W_2$ *where* $W_i \subseteq \mathbb{K}^{E_i}$.

Proof. The splitting of $\pi_i\colon \mathbb{K}^E \to \mathbb{K}^{E_i}$ allows one to consider $W_i := \pi_i(W) \subseteq \mathbb{K}^{E_i}$. Decompose a basis $B = B_1 \sqcup B_2 \in \mathcal{B}_{\mathsf{M}}$ into $B_i \in \mathcal{B}_{\mathsf{M}_i}$ where $i = 1, 2$. Then $\pi_i \circ \alpha_{W,B}$ factors through isomorphisms $\alpha_{W_i,B_i}\colon W_i \to \mathbb{K}^{B_i}$ where $i = 1, 2$. Composing with $\mathbb{K}^{B_i} \hookrightarrow \mathbb{K}^{E_i}$ shows that $W_i \subseteq W$ and hence $W_i = W \cap \mathbb{K}^{E_i}$ for $i = 1, 2$. It follows that $\mathbb{K}^E = \mathbb{K}^{E_1} \oplus \mathbb{K}^{E_2}$ induces $W = W_1 \oplus W_2$. $\square$

Proposition 3.10 (Connectedness and irreducibility)**.** *Let* M *be a matroid of rank* $\operatorname{rk}\mathsf{M} > 0$ *with realization* $W \subseteq \mathbb{K}^E$. *Then* M *is connected if and only if* M *has no loops and* ψ_W *is irreducible. In particular, if* $\mathsf{M} = \bigoplus_{i=1}^n \mathsf{M}_i$ *is a decomposition into connected components* M_i, *then* $\psi_W = \prod_{i=1}^n \psi_{W_i}$ *where* ψ_{W_i} *is irreducible if* $\operatorname{rk}\mathsf{M}_i > 0$, *and* $\psi_{W_i} = 1$ *otherwise*

Proof. First suppose that $\mathsf{M} = \mathsf{M}_1 \oplus \mathsf{M}_2$ is disconnected with underlying proper partition $E = E_1 \sqcup E_2$. By Lemma 3.9, any realization $W \subseteq \mathbb{K}^E$ of M decomposes as $W = W_1 \oplus W_2$ where $W_i \subseteq \mathbb{K}^{E_i}$. Then $\alpha_{W,B} = \alpha_{W_1,B_1} \oplus \alpha_{W_2,B_2}$ for all $B = B_1 \sqcup B_2 \in \mathcal{B}_{\mathsf{M}}$ and hence $\psi_W = \psi_{W_1} \cdot \psi_{W_2}$. This factorization is proper if M and hence each M_i has no loops (see Remark 3.5). Thus ψ_W is reducible in this case.

Suppose now that ψ_W is reducible for some realization $W \subseteq \mathbb{K}^E$ of M. Then $\psi_W = \psi_1 \cdot \psi_2$ with ψ_i homogeneous of positive degree, for $i = 1, 2$. Since ψ_W is a linear combination of square-free monomials (see Definition 3.2), this yields a proper partition $E = E_1 \sqcup E_2$ such that $\psi_i \in \mathbb{K}[x_{E_i}]$, for $i = 1, 2$. In particular, there is no cancellation of terms in the product $\psi_W = \psi_1 \cdot \psi_2$. Consider the corresponding restrictions $\mathsf{M}_i = \mathsf{M}|_{E_i}$, for $i = 1, 2$.

Each basis $B \in \mathcal{B}_{\mathsf{M}}$ indexes a monomial x^B of ψ_W. Set $B_i := B \cap E_i$, for $i = 1, 2$. Then $x^B = x^{B_1} \cdot x^{B_2}$ where x^{B_i} is a monomial of ψ_i, for $i = 1, 2$. By homogeneity of ψ_i, B_i is a basis of M_i, for $i = 1, 2$, and hence $B = B_1 \sqcup B_2 \in \mathcal{B}_{\mathsf{M}_1 \oplus \mathsf{M}_2}$. It follows that $\mathcal{B}_{\mathsf{M}} \subseteq \mathcal{B}_{\mathsf{M}_1 \oplus \mathsf{M}_2}$.

Conversely, let $B = B_1 \sqcup B_2 \in \mathcal{B}_{\mathsf{M}_1 \oplus \mathsf{M}_2}$. By definition, $B_i \in \mathcal{B}_{\mathsf{M}_i}$ is of the form $B_i = B \cap E_i$ for some $B \in \mathcal{B}_{\mathsf{M}}$, for $i = 1, 2$. As above, x^{B_i} is then a monomials in ψ_i, for $i = 1, 2$. Since there is no cancellation of terms in the product $\psi_W = \psi_1 \cdot \psi_2$, x^B is then a monomial of ψ_W, and hence $B \in \mathcal{B}_{\mathsf{M}}$. It follows that $\mathcal{B}_{\mathsf{M}} \supseteq \mathcal{B}_{\mathsf{M}_1 \oplus \mathsf{M}_2}$ as well.

So $\mathsf{M} = \mathsf{M}_1 \oplus \mathsf{M}_2$ is a proper decomposition, and M is disconnected. $\square$

We use the following well-known fact from linear algebra.

Remark 3.11 (Determinant formula). Consider a short exact sequence of finite dimensional $\mathbb{K}$-vector spaces

$$0 \longrightarrow W \longrightarrow V \longrightarrow U \longrightarrow 0.$$

Abbreviate $\bigwedge V := \bigwedge^{\dim V} V$. There is a unique isomorphism

$$\bigwedge W \otimes \bigwedge U = \bigwedge V \tag{3.5}$$

that fits into a commutative diagram of canonical maps

$$\begin{tikzcd} \bigwedge W \otimes \bigwedge^{\dim U} V \arrow[r] \arrow[d] & \bigwedge^{\dim W} V \otimes \bigwedge^{\dim U} V \arrow[d] \\ \bigwedge W \otimes \bigwedge U \arrow[r, equal] & \bigwedge V. \end{tikzcd}$$

Tensored with

$$(\bigwedge U)^\vee = \bigwedge (U^\vee), \quad (\bigwedge W)^\vee = \bigwedge (W^\vee)$$

respectively it induces identifications

$$\bigwedge W = \bigwedge V \otimes \bigwedge U^\vee, \quad \bigwedge U = \bigwedge W^\vee \otimes \bigwedge V. \tag{3.6}$$

Consider a commutative diagram with short exact rows

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r] \arrow[d, "\alpha"', "\cong"] & V \arrow[r] \arrow[d, equal] & U \arrow[r] & 0 \\ 0 & U' \arrow[l] & V \arrow[l] & W' \arrow[l] \arrow[u, "\cong", "\beta"'] & 0. \arrow[l] \end{tikzcd}$$

Applying (3.5) to the rows yields a composed isomorphism

$$\bigwedge \alpha \otimes \bigwedge \beta^{-1} \colon \bigwedge W \otimes \bigwedge U \longrightarrow \bigwedge U' \otimes \bigwedge W'.$$

and with (3.6) a commutative diagram

$$\begin{tikzcd}
\bigwedge W \arrow[r, equal] \arrow[d, "\bigwedge\alpha"', "\cong"] & \bigwedge W \otimes \bigwedge U \otimes \bigwedge U^\vee \arrow[r, equal] \arrow[d, "\bigwedge\alpha\otimes\bigwedge\beta^{-1}\otimes\bigwedge\beta^\vee"', "\cong"] & \bigwedge V \otimes \bigwedge U^\vee \arrow[d, "\cong"', "\mathrm{id}\otimes\bigwedge\beta^\vee"] \\
\bigwedge U' \arrow[r, equal] & \bigwedge U' \otimes \bigwedge W' \otimes \bigwedge W'^\vee \arrow[r, equal] & \bigwedge V \otimes \bigwedge W'^\vee.
\end{tikzcd}$$

◇

The following result describes the behavior of configuration polynomials under duality. The proof by Bloch, Esnault and Kreimer for graph polynomials applies verbatim (see [BEK06, Prop. 1.6]).

We consider $E^\vee$ as the dual basis of E and identify

$$(\mathbb{K}^E)^\vee = \mathbb{K}^{E^\vee}.$$

The bijection (2.5) extends to a $\mathbb{K}$-linear isomorphism

$$\nu\colon \mathbb{K}^E \to \mathbb{K}^{E^\vee}.$$

Proposition 3.12 (Dual configuration polynomial)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M. Then, for a suitable choice of c_W,*

$$\det \alpha_{W^\perp,S^\perp} = \pm \det \alpha_{W,S}$$

for all $S \subseteq E$ of size $|S| = \operatorname{rk}\mathsf{M}$. In particular,

$$\psi_{W^\perp} = x^E \cdot \psi_W((x_{e^\vee}^{-1})_{e\in E}).$$

Proof. Let $S \subseteq E$ be of size $|S| = \operatorname{rk}\mathsf{M}$. Then $S \in \mathcal{B}_{\mathsf{M}}$ if and only if $S^\perp \in \mathcal{B}_{\mathsf{M}^\perp}$. We may assume that this is the case as otherwise both determinants are zero. Then there a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] \arrow[d, "\alpha_{W,S}"', "\cong"] & \mathbb{K}^E \arrow[r] \arrow[d, "\cong"', "\nu"] & \mathbb{K}^E/W \arrow[r] & 0 \\
0 & \mathbb{K}^S \arrow[l] & \mathbb{K}^{E^\vee} \arrow[l, "\pi_S\circ\nu^{-1}"'] & \mathbb{K}^{S^\perp} \arrow[l, "\pi_{S^\perp}^\vee"'] \arrow[u, "\cong", "\alpha_{W^\perp,S^\perp}^\vee"'] & 0. \arrow[l]
\end{tikzcd}$$

This yields a commutative diagram (Remark 3.11)

$$\begin{tikzcd}
\mathbb{K} \arrow[rr, "\cong"'] \arrow[d, "c_W"'] & & \bigwedge^{|E|}\mathbb{K}^E \otimes_{\mathbb{K}} \mathbb{K} \arrow[d, "\mathrm{id}\otimes c_{W^\perp}"] \\
\bigwedge^{\operatorname{rk}\mathsf{M}} W \arrow[rr, equal] \arrow[d, "\bigwedge^{\operatorname{rk}\mathsf{M}}\alpha_{W,S}"'] & & \bigwedge^{|E|}\mathbb{K}^E \otimes \bigwedge^{\operatorname{rk}\mathsf{M}^\perp} W^\perp \arrow[d, "\bigwedge^{|E|}\nu\otimes\bigwedge^{\operatorname{rk}\mathsf{M}^\perp}\alpha_{W^\perp,S^\perp}"] \\
\bigwedge^{\operatorname{rk}\mathsf{M}}\mathbb{K}^S \arrow[rr, equal] & & \bigwedge^{|E|}\mathbb{K}^{E^\vee} \otimes \bigwedge^{\operatorname{rk}\mathsf{M}^\perp}\mathbb{K}^{S^\perp}.
\end{tikzcd}$$

Up to a sign, the composition

$$\mathbb{K} = \bigwedge^{|E|}\mathbb{K}^E \xrightarrow{\bigwedge^{|E|}\nu} \bigwedge^{|E|}\mathbb{K}^{E^\vee} = \mathbb{K}$$

is the identity. A suitable choice of c_W yields the claim (see Remark 3.3). ☐

The coefficients of the configuration polynomial satisfy the following restriction-contraction formula.

Lemma 3.13 (Restriction-contraction for coefficients)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M. For $B \in \mathcal{B}_{\mathsf{M}}$ and $F \subseteq E$, $B \cap F \in \mathcal{B}_{\mathsf{M}|_F}$ if and only if $B \backslash F \in \mathcal{B}_{\mathsf{M}/F}$. In this case,*

$$c_{W,B} = c_F^2 \cdot c_{W/F,B\backslash F} \cdot c_{W|_F,B\cap F}$$

where $c_F = c_{W/F}^{-1} \cdot c_{W|F}^{-1} \cdot c_W \in \mathbb{K}^*$ is independent of B.

Proof. The equivalence follows from the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & W/F \arrow[r] \arrow[d, hook] & W \arrow[r] \arrow[d, hook] & W|_F \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{K}^{E\backslash F} \arrow[r] \arrow[d] & \mathbb{K}^E \arrow[r] \arrow[d] & \mathbb{K}^F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{K}^{B\backslash F} \arrow[r] & \mathbb{K}^B \arrow[r] & \mathbb{K}^{B\cap F} \arrow[r] & 0.
\end{tikzcd}$$

Taking exterior powers yields (see Remark 3.11)

(3.7)

$$\begin{tikzcd}
\mathbb{K} \arrow[rr, "c_F", "\cong"'] \arrow[d, "c_W"', "\cong"] & & \mathbb{K} = \mathbb{K} \otimes \mathbb{K} \arrow[d, "\cong"', "c_{W/F} \otimes c_{W|_F}"] \\
\bigwedge^{\operatorname{rk} \mathsf{M}} W \arrow[rr, equal] \arrow[d, "\bigwedge^{\operatorname{rk}\mathsf{M}} \alpha_{W,B}"'] & & \bigwedge^{\operatorname{rk} \mathsf{M}/F} W/F \otimes \bigwedge^{\operatorname{rk} \mathsf{M}|_F} W|_F \arrow[d, "\bigwedge^{\operatorname{rk}\mathsf{M}/F} \alpha_{W/F,B\backslash F} \otimes \bigwedge^{\operatorname{rk}\mathsf{M}|_F} \alpha_{W|_F,B\cap F}"] \\
\bigwedge^{\operatorname{rk} \mathsf{M}} \mathbb{K}^B \arrow[rr, equal] & & \bigwedge^{\operatorname{rk} \mathsf{M}/F} \mathbb{K}^{B\backslash F} \otimes \bigwedge^{\operatorname{rk} \mathsf{M}|_F} \mathbb{K}^{B\cap F}
\end{tikzcd}$$

where $c_F = c_{W/F}^{-1} \cdot c_{W|F}^{-1} \cdot c_W \in \mathbb{K}^*$ is independent of B. ☐

The following result describes the behavior of configuration polynomials under deletion-contraction. The statement on $\partial_e \psi_W$ was proven by Patterson (see [Pat10, Lem. 4.4]).

Proposition 3.14 (Deletion-contraction for configuration polynomials)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M. Then*

$$\psi_W = \begin{cases} \psi_{W\backslash e} = \psi_{W/e} & \textit{if } e \textit{ is a loop}, \\ \psi_{W|_e} \cdot \psi_{W/e} = \psi_{W|_e} \cdot \psi_{W\backslash e} & \textit{if } e \textit{ is a coloop}, \\ \psi_{W\backslash e} + \psi_{W|_e} \cdot \psi_{W/e} & \textit{otherwise}, \end{cases}$$

where $\psi_{W|_e} = c_{W|_e,\{e\}} \cdot x_e$ *with* $c_{W|_e,\{e\}} \in \mathbb{K}^*$ *if* e *is not a loop. In particular,*

$$\partial_e \psi_W = \begin{cases} 0 & \text{if } e \text{ is a loop,} \\ \psi_{W/e} = \psi_{W\setminus e} & \text{if } e \text{ is a coloop,} \\ \psi_{W/e} & \text{otherwise,} \end{cases}$$

$$\psi_W|_{x_e=0} = \begin{cases} \psi_{W\setminus e} = \psi_{W/e} & \text{if } e \text{ is a loop,} \\ 0 & \text{if } e \text{ is a coloop,} \\ \psi_{W\setminus e} & \text{otherwise.} \end{cases}$$

Proof. Decompose

$$\psi_W = \sum_{e\notin B\in\mathcal{B}_{\mathsf{M}}} c_{W,B} \cdot x^B + x_e \cdot \sum_{e\in B\in\mathcal{B}_{\mathsf{M}}} c_{W,B} \cdot x^{B\setminus\{e\}}. \tag{3.8}$$

The second sum in (3.8) is non-zero if and only if e is not a loop. By Lemma 3.13 applied to $F = \{e\}$, it equals (see (2.3))

$$c^2 \cdot c_{W|_e,\{e\}} \cdot \sum_{B\in\mathcal{B}_{\mathsf{M}/e}} c_{W/e,B\setminus\{e\}} \cdot x^B = c^2 \cdot c_{W|_e,\{e\}} \cdot \psi_{W/e}$$

where $c := c_{\{e\}} \in \mathbb{K}^*$.

The first sum in (3.8) is non-zero if and only if e is not a coloop. In this case, $F := E\setminus\{e\}$ satisfies $W/F = W \cap \mathbb{K}^{\{e\}} = 0$ and $W|_F = W/(W \cap \mathbb{K}^{\{e\}}) = W$. It follows that $c_{W/F,\varnothing} = 1$ and (see (3.7))

$$\bigwedge^{\operatorname{rk}\mathsf{M}|_F} W|_F = \bigwedge^{\operatorname{rk}\mathsf{M}/e} W/e \otimes \bigwedge^{\operatorname{rk}\mathsf{M}|_e} W|_e$$

yields $c_F = c_{\{e\}} = c$. By Lemma 3.13, the first sum in (3.8) then equals (see (2.1))

$$c^2 \cdot \sum_{B\in\mathcal{B}_{\mathsf{M}\setminus e}} c_{W\setminus e,B} \cdot x^B = c^2 \cdot \psi_{W\setminus e}.$$

If e is a (co)loop, then $W/e = W\setminus e$ (see Remark 2.15.(a)). This yields the claimed formulas up to the factor c^2, but $c = 1$ for a suitable choice of c_W (see Remark 3.3). ☐

The following formula relates configuration polynomials with deletion and contraction of handles. It is the basis for our inductive approach to Jacobian schemes.

Corollary 3.15 (Configuration polynomials and handles). *Let* $W \subseteq \mathbb{K}^E$ *be a realization of a connected matroid* M *on* E*, and let* $E \neq H \in \mathcal{H}_{\mathsf{M}}$ *be a proper handle. Then* $H \in \mathcal{C}_{\mathsf{M}/(E\setminus H)}$ *and*

$$\psi_W = \psi_{W/(E\setminus H)} \cdot \psi_{W\setminus H} + \psi_{W|_H} \cdot \psi_{W/H}, \tag{3.9}$$

$$\psi_{W/(E\setminus H)} = \sum_{h\in H} \psi_{W|_{H\setminus\{h\}}}, \tag{3.10}$$

$$\psi_{W|_H} = x^H, \quad \psi_{W|_{H\setminus\{h\}}} = x^{H\setminus\{h\}}. \tag{3.11}$$

In particular, after suitably scaling H,

$$\psi_W = \sum_{h\in H} x^{H\setminus\{h\}} \cdot \psi_{W\setminus H} + x^H \cdot \psi_{W/H}. \tag{3.12}$$

Proof. By Lemma 2.3.(b), there is a $H \subsetneq C \in \mathcal{C}_{\mathsf{M}}$. Since $H \in \mathcal{H}_{\mathsf{M}}$, $H \subseteq C$ for any $C' \in \mathcal{C}_{\mathsf{M}}$ with $C' \nsubseteq E\setminus H$. This yields the first claim (see (2.4)) and hence (3.10) by Example 3.7.(b). By Lemma 2.3.(b) (see (2.1)), $\mathsf{M}|_H$ is free, and equalities (3.11) follows from Example 3.7.(a). Equality (3.12) follows from (3.9), (3.10) and Example 3.7.(b). It remains to prove equality (3.9).

We proceed by induction on $|H|$. Proposition 3.14 covers the case $|H| = 1$. Suppose now $|H| \geqslant 2$. Let $h \in H$ and set $H' := H\setminus\{h\}$. Since M is connected,

$$\psi_W = \psi_{W\setminus h} + \psi_{W|_h} \cdot \psi_{W/h} \tag{3.13}$$

by Proposition 3.14. By Lemma 2.3.(c) and (b), the set H' consists of coloops in $\mathsf{M}\setminus h$ and $\mathsf{M}|_{H'}$ is free. Iterating Proposition 3.14 thus yields

$$\psi_{W\setminus h} = \prod_{h'\in H'} \psi_{W|_{h'}} \cdot \psi_{W\setminus H} = \psi_{W|_{H'}} \cdot \psi_{W\setminus H}. \tag{3.14}$$

By Lemma 2.3.(d), the set H' is a proper handle in the connected matroid M/h. By Lemma 2.3.(c), h is a coloop in $\mathsf{M}\setminus H'$ and hence

$$W/h\setminus H' = W\setminus H'/h = W\setminus H'\setminus h = W\setminus H.$$

by Remark 2.15.(a). By the induction hypothesis,

$$\psi_{W/h} = \sum_{h'\in H'} \psi_{W|_{H'\setminus\{h'\}}} \cdot \psi_{W\setminus H} + \psi_{W|_{H'}} \cdot \psi_{W/H}. \tag{3.15}$$

By Lemma 2.3.(b), $\mathsf{M}|_H$ and $\mathsf{M}|_{H\setminus\{h'\}}$ are free. Iterating Proposition 3.14 thus yields

$$\psi_{W|_h} \cdot \psi_{W|_{H'}} = \psi_{W|_H}, \quad \psi_{W|_h} \cdot \psi_{W|_{H'\setminus\{h'\}}} = \psi_{W|_{H\setminus\{h'\}}}, \tag{3.16}$$

by Proposition 3.14. Using equalities (3.10) and (3.16), equality (3.9) is obtained by substituting (3.14) and (3.15) into (3.13). ☐

The following result describes the behavior of configuration polynomials when passing to a hyperplane. It is not needed to prove our main result.

Proposition 3.16 (Configuration polynomial of hyperplanes). *Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M, and let $0 \neq \varphi \in W^\vee$. Then*

$$\psi_{W_\varphi} = \sum_{\substack{B\subseteq E\\ |B|=\operatorname{rk}\mathsf{M}-1}} \left(\sum_{e\notin B} \pm\tilde{\varphi}_e \cdot \det \alpha_{W,B\cup\{e\}}\right)^2 x^B,$$

where $\tilde{\varphi} = (\tilde{\varphi}_e)_{e\in E} \in (\mathbb{K}^E)^\vee$ is any lift of φ.

Proof. Set $V := W^\perp$ and $V_\varphi := W_\varphi^\perp$ and consider the commutative diagram with short exact rows and columns

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \mathbb{K} \arrow[d] & \\
0 \arrow[r] & W_\varphi \arrow[r] \arrow[d] & \mathbb{K}^E \arrow[r] \arrow[d, equal] & V_\varphi^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W \arrow[r] \arrow[d, "\varphi"'] & \mathbb{K}^E \arrow[r] \arrow[dl, "\tilde\varphi"] & V^\vee \arrow[r] \arrow[d] & 0 \\
 & \mathbb{K} \arrow[d] & & 0 & \\
 & 0. & & &
\end{tikzcd}$$

Dualizing and identifying the two copies of $\mathbb{K}$ by the Snake Lemma yields a commutative diagram with short exact rows and columns

(3.17)
$$\begin{tikzcd}
 & & & 0 & \\
 & 0 & & \mathbb{K} \arrow[u] \arrow[dl, "\cdot\tilde\varphi"'] & \\
0 & W_\varphi^\vee \arrow[l] \arrow[u] & \mathbb{K}^{E^\vee} \arrow[l] & V_\varphi \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & W^\vee \arrow[l] \arrow[u] & \mathbb{K}^{E^\vee} \arrow[l] \arrow[u, equal] & V \arrow[l] \arrow[u] & 0 \arrow[l] \\
 & \mathbb{K} \arrow[u, "\cdot\varphi"] \arrow[ur, "\cdot\tilde\varphi"'] & & 0 \arrow[u] & \\
 & 0. \arrow[u] & & &
\end{tikzcd}$$

By Remark 3.11 and with a suitable choice of c_V (see Remark 3.3), the right vertical short exact sequence in (3.17) gives rise to a commutative square

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "c_{V_\varphi}"] \arrow[d, equal] & \bigwedge^{\operatorname{rk}\mathsf{M}^\perp+1} V_\varphi \arrow[d, equal] \\
\mathbb{K} \arrow[r, "c_V"] & \bigwedge^{\operatorname{rk}\mathsf{M}^\perp} V
\end{tikzcd}$$

Let $B' \subseteq E^\vee$ with $|B'| = \dim V_\varphi = \operatorname{rk} \mathsf{M}^\perp + 1$ and denote $\tilde\varphi_{B'} = (\tilde\varphi_e)_{e \in B'}$. Due to (3.17) the maps $\alpha_{V_\varphi, B'}$ and

$$\begin{pmatrix} \tilde\varphi_{B'} & \alpha_{V,B'} \end{pmatrix} : \mathbb{K} \oplus V \to \mathbb{K}^{E^\vee} \to \mathbb{K}^{B'}$$

agree after applying $\bigwedge^{\operatorname{rk} \mathsf{M}^\perp + 1}$. Laplace expansion thus yields

$$\det \alpha_{V_\varphi, B'} = \sum_{e \in B'} \pm \tilde\varphi_e \cdot \det \alpha_{V, B' \setminus \{e\}}.$$

Let $B \subseteq E$ with $|B| = \dim W_\varphi = \operatorname{rk} \mathsf{M} - 1$ and $B' = B^\perp$. Then Proposition 3.12 yields

$$c_{W_\varphi, B} = \left(\sum_{e \notin B} \pm \tilde\varphi_e \cdot \det \alpha_{W, B \cup \{e\}} \right)^2. \qquad \square$$

3.2. **Graph polynomials.** We continue the discussion of graphic matroids from §2.4 discussing their configuration polynomials.

Let $G = (E, V)$ be a graph.

Definition 3.17 (Graph polynomial)**.** The *(first) graph polynomial* or *Kirchhoff polynomial* of a graph G is the polynomial

$$\psi_G := \sum_{T \in \mathcal{T}(G)} x^T.$$

By (2.7), we have $\psi_G = \psi_W$ for any totally unimodular realization W of $\mathsf{M}(G)$. In particular, this yields the following result of Bloch, Esnault and Kreimer (see [BEK06, Prop. 2.2] and Proposition 3.12).

Proposition 3.18 (Bloch, Esnault, Kreimer)**.** *For any graph G, we have (see Definition 2.17)*

$$\psi_G = \psi_{W_G}. \qquad \square$$

Denote by $\mathcal{T}_2(G)$ the set of acyclic subgraphs T of G with $|V| - 2$ edges. Any such T has 2 connected components T_1 and T_2 and we write $T = \{T_1, T_2\}$. For any subgraph S of G and $p \in \mathbb{K}^V$ we abbreviate

$$m_S(p) := \sum_{v \in S} p_v.$$

If $p \in \ker \sigma$ (see (2.8)) and $T \in \mathcal{T}_2(G)$, then

$$m_{T_1}(p) = \sum_{v \in T_1} p_v = - \sum_{v \in T_2} p_v = -m_{T_2}(p)$$

and hence $m_{T_1}^2(p) \in \mathbb{K}$ is well-defined.

Definition 3.19 (Second graph polynomial)**.** The *second graph polynomial* of a graph G over $\mathbb{K}$ is the polynomial

$$\psi_G(p) := \sum_{\{T_1, T_2\} \in \mathcal{T}_2(G)} m_{T_1}^2(p) \cdot x^{T_1 \sqcup T_2}$$

depending on a *momentum* $0 \neq p \in \ker \sigma$ for G over $\mathbb{K}$.

The following is a reformulation of a result of Patterson realizing the second graph polynomial as a configuration polynomial of hyperplanes (see [Pat10, Prop. 3.3]). Patterson's proof makes the general formula in Proposition 3.16 explicit in case of graph configurations (see [Pat10, Lem. 3.4]).

Proposition 3.20 (Patterson)**.** *For any graph G and momentum p of G over $\mathbb{K}$, we have (see Definitions 3.19, 2.14.*(b) *and 2.17)*

$$\psi_G(p) = \psi_{(W_G)_p}. \qquad \square$$

3.3. **Configuration form.** The configuration form yields an equivalent definition of the configuration polynomial. Its second degeneracy scheme turn out to be closely related to the Jacobian scheme of non-smooth points of the hypersurface defined by the corresponding configuration polynomial.

Definition 3.21 (Configuration form)**.** Let $\mu_{\mathbb{K}}$ denote the multiplication map of $\mathbb{K}$. Consider the generic diagonal bilinear form on $\mathbb{K}^E$,

$$Q := \sum_{e\in E} x_e \cdot \mu_{\mathbb{K}} \circ (e^\vee \times e^\vee) \colon \mathbb{K}^E \times \mathbb{K}^E \to \mathbb{K}[x].$$

Let $W \subseteq \mathbb{K}^E$ be a configuration of rank $r = \dim_{\mathbb{K}} W$. Then the *configuration (bilinear) form* of W is the restriction of Q to W,

$$Q_W := Q|_{W\times W} \colon W \times W \to \mathbb{K}[x].$$

Alternatively, it can be considered as the composition of canonical maps

$$Q_W \colon W[x] \longrightarrow \mathbb{K}^E[x] \xrightarrow{Q} \mathbb{K}^{E^\vee}[x] \longrightarrow W^\vee[x], \tag{3.18}$$

where $-[x]$ means $-\otimes \mathbb{K}[x]$. For $k = 0, \dots, r$, it defines a map

$$\bigwedge^{r-k} W \otimes \bigwedge^{r-k} W \otimes \mathbb{K}[x] \to \mathbb{K}[x].$$

Its image is the kth Fitting ideal $\mathrm{Fitt}_k \operatorname{coker} Q_W$ (see [Eis95, §20.2]) and defines the $k-1$*st degeneracy scheme* of Q_W. We set

$$M_W := \mathrm{Fitt}_1 \operatorname{coker} Q_W \trianglelefteq \mathbb{K}[x].$$

Remark 3.22 (Matrix representation of configuration forms)*.* With respect to a basis $w = (w^1, \dots, w^r)$ of W, Q_W becomes a matrix of Hadamard products

$$Q_w = (x \star w^i \star w^j)_{i,j} = \left(\sum_{e\in E} x_e \cdot w^i_e \cdot w^j_e\right)_{i,j} \in \mathbb{K}^{r\times r}, \quad w^i_e := e^\vee(w^i).$$

Let $Q(i,j)$ denote the submaximal minor of a square matrix Q obtained by deleting row i and column j. Then

$$M_W = \langle Q_W(i,j) \mid i,j \in \{1,\dots,r\}\rangle.$$

Any basis of W can be written as $w' = Uw$ for some $U \in \operatorname{Aut}_{\mathbb{K}} W$. Then

$$Q_{w'} = UQ_wU^t.$$

and the $Q_{w'}(i,j)$ become $\mathbb{K}$-linear combinations of the $Q_w(i,j)$. We often consider Q_W as a matrix Q_w determined up to conjugation. ◇

Remark 3.23 (Configuration forms and basis scaling). Scaling E results in scaling of x in Q and in M_W (see Remark 3.4). ◇

Bloch, Esnault and Kreimer defined ψ_W in terms of Q_W (see [BEK06, Lem. 1.3]).

Lemma 3.24 (Configuration polynomial and form). *For any configuration $W \subseteq \mathbb{K}^E$, there is an equality $\psi_W = \det Q_W$.* □

The following result describes the behavior of Fitting ideals of configuration forms under duality. We consider the torus

$$\mathbb{T}^E := (\mathbb{K}^*)^E \subset \mathbb{K}^E.$$

We glue $\mathbb{K}^E$ and $\mathbb{K}^{E^\vee}$ along their tori by identifying

$$\mathbb{T}^E = \mathbb{T}^{E^\vee}, \quad x_e^{-1} = x_{e^\vee}, \quad e \in E.$$

Proposition 3.25 (Duality of cokernels of configuration forms). *Let $W \subseteq \mathbb{K}^E$ be a configuration. Then there is an isomorphism of $\mathbb{K}[\mathbb{T}^E]$-modules*

$$\operatorname{coker}(Q_W)_{x^E} \cong \operatorname{coker}(Q_{W^\perp})_{x^{E^\vee}},$$

where the lower index denotes localization. In particular,

$$(M_W)_{x^E} = (M_{W^\perp})_{x^{E^\vee}} \trianglelefteq \mathbb{K}[\mathbb{T}^E].$$

Proof. Consider the short exact sequence

$$0 \longrightarrow W \longrightarrow \mathbb{K}^E \longrightarrow \mathbb{K}^E/W \longrightarrow 0 \tag{3.19}$$

and its $\mathbb{K}$-dual

$$0 \longleftarrow W^\vee \longleftarrow \mathbb{K}^{E^\vee} \longleftarrow W^\perp \longleftarrow 0. \tag{3.20}$$

We identify $\mathbb{K}^E = \mathbb{K}^{E^{\vee\vee}}$ and $\mathbb{K}^E/W = W^{\perp\vee}$, and we abbreviate

$$Q^\vee := Q_{\mathbb{K}^{E^\vee}}.$$

Then Q_{x^E} and $Q^\vee_{x^{E^\vee}}$ are mutual inverses by definition. Together with (3.19) and (3.20) tensored by $\mathbb{K}[x^{\pm 1}]$ and (3.18) for W and $W^\perp$, they

fit into commutative diagram with exact rows and columns,
(3.21)

$$
\begin{tikzcd}
 & & & 0 & \\
 & 0 \arrow[d] & & \operatorname{coker}(Q_{W^\perp})_{x^{E^\vee}} \arrow[u] & \\
0 \arrow[r] & W[x^{\pm1}] \arrow[r] \arrow[d, "(Q_W)_{x^E}"'] & \mathbb{K}^E[x^{\pm1}] \arrow[r] \arrow[d, shift right, "Q_{x^E}"'] & W^{\perp\vee}[x^{\pm1}] \arrow[r] \arrow[u] & 0 \\
0 & W^\vee[x^{\pm1}] \arrow[l] \arrow[d] & \mathbb{K}^{E^\vee}[x^{\pm1}] \arrow[l] \arrow[u, shift right, "Q^\vee_{x^{E^\vee}}"'] & W^\perp[x^{\pm1}] \arrow[l] \arrow[u, "(Q_{W^\perp})_{x^{E^\vee}}"'] & 0 \arrow[l] \\
 & \operatorname{coker}(Q_W)_{x^E} \arrow[d] & & 0 \arrow[u] & \\
 & 0, & & &
\end{tikzcd}
$$

where $-[x^{\pm1}]$ means $-\otimes \mathbb{K}[x^{\pm1}]$. Injectivity of $(Q_W)_{x^E}$, and similarly of $(Q_{W^\perp})_{x^{E^\vee}}$, comes from $\det(Q_W)_{x^E} = \psi_W \in \mathbb{K}[x^{\pm1}]$ being regular if $W \neq 0$ (see Lemma 3.24 and Remark 3.5). The claim follows from the diagram (3.21) using the universal property of cokernels. $\square$

The following result describes the behavior of submaximal minors of configuration forms under deletion-contraction. It is the basis for our inductive approach to second degeneracy schemes.

Lemma 3.26 (Deletion-contraction for submaximal minors)**.** *Let $W \subseteq \mathbb{K}^E$ be a configuration of rank $r = \dim_{\mathbb{K}} W$ and $e \in E$. Then any basis of W/e can be extended to bases of W and $W\backslash e$ such that $Q_W(i,j) =$*

$$
\begin{cases}
Q_{W\backslash e}(i,j) = Q_{W/e}(i,j) & \text{if } e \text{ is a loop,}\\
\psi_{W\backslash e} = \psi_{W/e} & \text{if } e \text{ is a coloop, } i = r = j,\\
x_e \cdot Q_{W\backslash e}(i,j) = x_e \cdot Q_{W/e}(i,j) & \text{if } e \text{ is a coloop, } i \neq r \neq j,\\
0 & \text{if } e \text{ is a coloop, otherwise,}\\
\psi_{W/e} & \text{if } e \text{ is not a (co)loop, } i = r = j,\\
Q_{W\backslash e}(i,j) & \text{if } e \text{ is not a (co)loop, } i = r \text{ or } j = r,\\
Q_{W\backslash e}(i,j) + x_e \cdot Q_{W/e}(i,j) & \text{if } e \text{ is not a (co)loop, } i \neq r \neq j
\end{cases}
$$

for all $i, j \in \{1, \dots, r\}$. In particular, the $Q_W(i,j)$ are linear combinations of square-free monomials for any basis of W.

Proof. Pick a basis $w^1, \dots, w^r$ of $W \subseteq \mathbb{K}^E$ and consider

$$
Q_W = \left(\sum_{e \in E} x_e \cdot w_e^i \cdot w_e^j \right)_{i,j}
$$

as a matrix. Recall that (see Definition 2.14.(d) and (e)),

$$W\backslash e = \pi_{E\backslash\{e\}}(W), \quad W/e = W \cap \mathbb{K}^{E\backslash\{e\}}.$$

and the description of (co)loops in Remark 2.15.(a):

- If e is a loop, then $w_e^i = 0$ for all $i = 1,\dots,r$ and hence $W\backslash e = W = W/e$.
- If e is not a loop, then we may adjust $w^1,\dots,w^r$ such that $w_e^i = \delta_{i,r}$ for all $i = 1,\dots,r$ and then $w^1,\dots,w^{r-1}$ is a general basis of W/e.
- If e is a coloop, then we may adjust $w^r = e$ and $\pi_{E\backslash\{e\}}$ identifies $w^1,\dots,w^{r-1}$ with a basis of $W\backslash e = W/e$.

In the latter case,

$$Q_W = \begin{pmatrix} Q_{W\backslash e} & 0 \\ 0 & x_e \end{pmatrix}, \tag{3.22}$$

and the claim follows by Lemma 3.24.

It remains to consider the case in which e is not a (co)loop. Then $\iota_{E\backslash\{e\}}$ and $\pi_{E\backslash\{e\}}$ identify $w^1,\dots,w^{r-1}$ and $w^1,\dots,w^r$ with bases of W/e and $W\backslash e$ respectively. Hence,

$$Q_{W\backslash e} = \begin{pmatrix} Q_{W/e} & b \\ b^t & a \end{pmatrix}, \quad Q_W = \begin{pmatrix} Q_{W/e} & b \\ b^t & x_e + a \end{pmatrix} \tag{3.23}$$

where both the entry a and column b are independent of x_e. We consider two cases. If $i = r$ or $j = r$, then clearly $Q_W(i,j) = Q_{W\backslash e}(i,j)$. Otherwise,

$$Q_W(i,j) = Q_{W\backslash e}(i,j) + x_e \cdot Q_{W/e}(i,j).$$

This proves the claimed equalities and the particular claim follows (see Remark 3.22) ☐

As an application of Lemma 3.24 we describe the behavior of configuration polynomials under 2-separations.

Proposition 3.27 (Configuration polynomials and 2-separations)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a connected matroid M. If $E = E_1 \sqcup E_2$ is an (exact) 2-separation, then*

$$\psi_W = \psi_{W/E_1} \cdot \psi_{W|_{E_1}} + \psi_{W|_{E_2}} \cdot \psi_{W/E_2}.$$

Proof. Adopt the notation of [Tru92, §8.2]. Extend a basis $B_2 \in \mathcal{B}_{\mathsf{M}|_{E_2}}$ to a basis $B \in \mathcal{B}_{\mathsf{M}}$. Then W is represented as the row space of a matrix (see [Tru92, (8.1.1)])

$$\begin{pmatrix} I & 0 & A_1 & 0 \\ 0 & I & D & A_2' \end{pmatrix} \tag{3.24}$$

where the block columns are indexed by $B\backslash B_2, B_2, E_1\backslash B \cap E_1, E_2\backslash B_2$ and $\operatorname{rk} D = 1$. After ordering and scaling B_2 and $E_1\backslash B \cap E_1$ suitably we may assume that

$$\begin{aligned} D &= (1\ b)^t a_1, \\ a_1 &= \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} \neq 0, \\ b &= \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}. \end{aligned}$$

The size of b and a_1 is determined by number of rows and columns of D, respectively. While b could be 0, at least one entry of a_1 is a 1. After suitable row operations and adjusting signs of x_{B_2}, the matrix (3.24) of W can be repartitioned as follows

$$\begin{pmatrix} I & 0 & 0 & A_1 & 0 \\ 0 & 1 & 0 & a_1 & a_2 \\ 0 & b^t & I & 0 & A_2 \end{pmatrix}. \tag{3.25}$$

Let $e \in E$ the index of the column $(0\ 1\ b)^t$. Let $X_1, x_e, X_2, X_1', X_2'$ be diagonal matrices of variables corresponding to the block columns of the matrix (3.25) of W. Then the corresponding matrix of Q_W takes the form

$$Q_W = \begin{pmatrix} X_1 + A_1 X_1' A_1^t & A_1 X_1' a_1^t & 0 \\ a_1 X_1' A_1^t & x_e + a_1 X_1' a_1^t + a_2 X_2' a_2^t & x_e b + a_2 X_2' A_2^t \\ 0 & b^t x_e + A_2 X_2' a_2^t & b^t x_e b + X_2 + A_2 X_2' A_2^t \end{pmatrix}.$$

It involves the matrices

$$\begin{aligned} Q_{W|_{E_1}} &= \begin{pmatrix} Q_{W/E_2} & A_1 X_1' a_1^t \\ a_1 X_1' A_1^t & a_1 X_1' a_1^t \end{pmatrix}, \\ Q_{W/E_2} &= X_1 + A_1 X_1' A_1^t, \\ Q_{W|_{E_2}} &= \begin{pmatrix} x_e + a_2 X_2' a_2^t & x_e b + a_2 X_2' A_2^t \\ b^t x_e + A_2 X_2' a_2^t & Q_{W/E_1} \end{pmatrix}, \\ Q_{W/E_1} &= b^t x_e b + X_2 + A_2 X_2' A_2^t. \end{aligned}$$

Laplace expansion of $\psi_W = \det Q_W$ along the eth column yields the claimed formula. $\square$

Remark 3.28. Let $W \subseteq \mathbb{K}^E$ be a realization of a connected matroid M, and let $H \in \mathcal{H}_{\mathsf{M}}$ be a separating handle. By Lemma 2.3.(e), $H \sqcup (E \backslash H)$ is a 2-separation of M. Proposition 3.27 applied to $(E_1, E_2) := (E \backslash H, H)$ thus yields the statement of Corollary 3.15 in this case. $\diamond$

Remark 3.29. Note that

$$\begin{aligned} d_1 &:= \deg \psi_{W|_{E_1}} = \deg \psi_{W/E_2} + 1, \\ d_2 &:= \deg \psi_{W|_{E_2}} = \deg \psi_{W/E_1} + 1. \end{aligned}$$

For $F \subseteq E$, consider the Euler operator $\chi_F = \sum_{e\in F} x_e\partial_e$. Then

$$\begin{aligned}\chi_{E_1}\psi_W &= d_1\psi_{W|_{E_1}}\psi_{W/E_1} + (d_1-1)\psi_{W/E_2}\psi_{W|_{E_2}},\\ \chi_{E_2}\psi_W &= (d_2-1)\psi_{W|_{E_1}}\psi_{W/E_1} + d_2\psi_{W/E_2}\psi_{W|_{E_2}},\end{aligned}$$

and subtracting respectively $d_1\psi_W$ and $d_2\psi_W$ yields

$$\psi_{W|_{E_1}}\psi_{W/E_1},\ \psi_{W/E_2}\psi_{W|_{E_2}} \in J_W.$$

So any prime over J_W contains a factor from each summand of ψ_W in the formula of Proposition 3.27. $\diamond$

4. Configuration hypersurfaces

In this section we establish our main results on Jacobian and second degeneracy schemes of realizations of connected matroids: The second degeneracy scheme is Cohen–Macaulay, the Jacobian scheme equidimensional, of codimension 3 (see Theorem 4.23). The second degeneracy scheme is reduced, the Jacobian scheme generically reduced if $\operatorname{ch}\mathbb{K} \neq 2$ (see Theorem 4.23).

4.1. **Commutative ring basics.** In this subsection we review preliminaries on equidimensionality and graded Cohen–Macaulayness. For the benefit of the non-experts we provide full proofs. Further we relate generic reducedness for a ring and an associated graded ring (see Lemma 4.5).

4.1.1. *Equidimensionality of rings.* Let R be a Noetherian ring. It is *equidimensional* if it is catenary and

$$\forall \mathfrak{p} \in \operatorname{Min}\operatorname{Spec} R\colon \forall \mathfrak{m} \in \operatorname{Max}\operatorname{Spec} R\colon \mathfrak{p} \subseteq \mathfrak{m} \implies \operatorname{height}(\mathfrak{m}/\mathfrak{p}) = \dim R.$$

In case R is an affine $\mathbb{K}$-algebra these two conditions reduce to (see [BH93, Thm. 2.1.12] and [Mat89, Thm. 5.6])

$$\forall \mathfrak{p} \in \operatorname{Min}\operatorname{Spec} R\colon\ \dim(R/\mathfrak{p}) = \dim R.$$

We say that R is *pure-dimensional* if

$$\forall \mathfrak{p} \in \operatorname{Ass} R\colon\ \dim(R/\mathfrak{p}) = \dim R.$$

The following lemma applies to any equidimensional affine $\mathbb{K}$-algebra.

Lemma 4.1. *Let R be a Noetherian ring such that $R_\mathfrak{m}$ is equidimensional for all $\mathfrak{m} \in \operatorname{Max}\operatorname{Spec} R$.*

(a) All saturated chains of primes in $\mathfrak{p} \in \operatorname{Spec} R$ have length $\operatorname{height}\mathfrak{p}$.
(b) For any $\mathfrak{p} \in \operatorname{Spec} R$, $x \in R$ and $\mathfrak{q} \in \operatorname{Spec} R$ minimal over $\mathfrak{p} + \langle x\rangle$,

$$\operatorname{height}\mathfrak{q} \leqslant \operatorname{height}\mathfrak{p} + 1.$$

Proof.

(a) Take two such chains of length n and n' starting at minimal primes $\mathfrak{p}_0$ and $\mathfrak{p}_0'$ respectively. Extend both by a saturated chain of primes of length m containing $\mathfrak{p}$ ending in a maximal ideal $\mathfrak{m}$. Since $R_\mathfrak{m}$ is equidimensional by hypothesis, the extended chains have length $n + m = n' + m$.

(b) By the Krull principal ideal theorem, $\operatorname{height}(\mathfrak{q}/\mathfrak{p}) \leqslant 1$. Take a chain of primes in $\mathfrak{p}$ of length $\operatorname{height}\mathfrak{p}$ and extend it by $\mathfrak{q}$ if $\mathfrak{p} \neq \mathfrak{q}$. By (a), it has length $\operatorname{height}\mathfrak{q}$ and the claim follows. □

Lemma 4.2. *Let R be an equidimensional affine $\mathbb{K}$-algebra and $x \in R$. If $R_x \neq 0$, then R_x is equidimensional of dimension $\dim R_x = \dim R$.*

Proof. Any minimal prime of R_x is of the form $\mathfrak{p}_x$ where $\mathfrak{p} \in \operatorname{Min}\operatorname{Spec} R$ with $x \notin \mathfrak{p}$. By a version of the Hilbert Nullstellensatz, $\bigcap \operatorname{Max} V(\mathfrak{p}) = \mathfrak{p}$ (see [Mat89, Thm. 5.5]). This yields an $\mathfrak{m} \in \operatorname{Max}\operatorname{Spec} R$ such that $\mathfrak{p} \subseteq \mathfrak{m} \not\ni x$. In particular $\mathfrak{p}_x \subseteq \mathfrak{m}_x \in \operatorname{Max}\operatorname{Spec} R_x$ and $\dim R_x/\mathfrak{p}_x = \operatorname{height}(\mathfrak{m}_x/\mathfrak{p}_x) = \operatorname{height}(\mathfrak{m}/\mathfrak{p}) = \dim R$. □

4.1.2. *Generic reducedness.* A Noetherian ring R is *generically reduced* if $R_\mathfrak{p}$ is reduced for all minimal primes $\mathfrak{p} \in \operatorname{Min}\operatorname{Spec} R$. Equivalently R satisfies *Serre's condition* R_0 that $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \in \operatorname{Min}\operatorname{Spec} R$. We use the same notions for the associated affine scheme $\operatorname{Spec} R$.

Definition 4.3 (Generic reducedness)**.** We call a Noetherian scheme X *generically reduced (or R_0) along a subscheme Y* if X is reduced at all generic points specializing to a point of Y. If $X = \operatorname{Spec} R$ is an affine scheme, then we use the same notions for the Noetherian ring R.

Lemma 4.4 (Reducedness and reduction)**.** *Let $(R, \mathfrak{m})$ be a local Noetherian ring. If R/tR is reduced for some parameter system t, then R is regular.*

Proof. By hypothesis, R/tR is local Artinian with maximal ideal $\mathfrak{m}/tR$. Reducedness makes R/tR a field and hence $\mathfrak{m} = tR$. Then R is regular by definition. □

Lemma 4.5 (R_0 and normal cone)**.** *Let R be a Noetherian d-dimensional ring and $I \trianglelefteq R$ an ideal. Consider the (extended) Rees $R[t]$-algebra (see [HS06, Def. 5.1.1])*

$$S := \operatorname{Rees}_I R = R[t, It^{-1}] \subseteq R[t^{\pm 1}]$$

and the associated graded ring $\bar{R} := \operatorname{gr}_I R = S/tS$.

(a) Suppose R is an equidimensional affine $\mathbb{K}$-algebra. Then S is a $d+1$-equidimensional affine $\mathbb{K}$-algebra. If in addition $I \neq R$, then $\bar{R}$ is a d-equidimensional affine $\mathbb{K}$-algebra.

(b) If S is equidimensional and $\bar{R}$ is R_0, then R is R_0 along $V(I)$.

Proof. There are ring homomorphisms

$$R \to R[t] \to S \to S/tS \cong \bar{R}.$$

Since R is Noetherian, I is finitely generated. Then S is a finite type R-algebra. In particular both S and $\bar{R}$ are Noetherian.

(a) Both Rees and gr commute with base change. After base change to $R/\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Min}\operatorname{Spec} R$ we may assume that R is a d-dimensional domain. Then S is a $(d+1)$-dimensional domain (see [HS06, Thm. 5.1.4]). Since R is an affine $\mathbb{K}$-algebra, so is S. In particular it is $(d+1)$-equidimensional. If $I \neq R$, then t is an S-sequence. With the Krull principal ideal theorem it follows that $S/tS \cong \bar{R}$ is d-equidimensional.

(b) Let $\mathfrak{p} \in \operatorname{Min}\operatorname{Spec} R$ be a minimal prime and consider the extension $\mathfrak{p}[t^{\pm 1}] \in \operatorname{Spec} R[t^{\pm 1}]$. Then (see [HS06, p. 96])

$$t \notin \tilde{\mathfrak{p}} := \mathfrak{p}[t^{\pm 1}] \cap S \in \operatorname{Min}\operatorname{Spec} S$$

and hence

$$S_{\tilde{\mathfrak{p}}} = (S_t)_{\tilde{\mathfrak{p}}_t} = R[t^{\pm 1}]_{\mathfrak{p}[t^{\pm 1}]}. \tag{4.1}$$

Since $\mathfrak{p}[t^{\pm 1}] \cap R = \mathfrak{p}$ the map $R \to R[t^{\pm 1}]$ localizes to an inclusion

$$R_{\mathfrak{p}} \hookrightarrow R[t^{\pm 1}]_{\mathfrak{p}[t^{\pm 1}]}. \tag{4.2}$$

To check injectivity, suppose $R_{\mathfrak{p}} \ni x/1 \mapsto 0 \in R[t^{\pm 1}]_{\mathfrak{p}[t^{\pm 1}]}$. Then $0 = xy \in R[t^{\pm 1}]$ for some $y = \sum_i y_i t^i \in R[t^{\pm 1}] \backslash \mathfrak{p}[t^{\pm 1}]$. Then $0 = xy_i \in R$ for all i and $y_j \in R \backslash \mathfrak{p}$ for some j. It follows that $0 = x/1 \in R_{\mathfrak{p}}$. Combining (4.1) and (4.2) reducedness of $R_{\mathfrak{p}}$ follows from reducedness of $S_{\tilde{\mathfrak{p}}}$.

Suppose now that $V(\mathfrak{p}) \cap V(I) \neq \varnothing$ and hence

$$R \neq \mathfrak{p} + I = \tilde{\mathfrak{p}}_0 + (tS)_0 = (\tilde{\mathfrak{p}} + tS)_0$$

implies $\tilde{\mathfrak{p}} + tS \neq S$. Let $\mathfrak{q} \in \operatorname{Spec} S$ be a minimal prime over $\tilde{\mathfrak{p}} + tS$. Then $\operatorname{height} \mathfrak{q} = 1$ by Lemma 4.1.(b) and since t is an S-sequence. Hence $\mathfrak{q}$ is minimal over tS and t is a parameter of $S_{\mathfrak{q}}$. Under $S/tS \cong \bar{R}$ the minimal prime $\mathfrak{q}/tS \in \operatorname{Spec}(S/tS)$ corresponds to a minimal prime $\bar{\mathfrak{q}} \in \operatorname{Spec} \bar{R}$. If $\bar{R}$ is R_0, then $S_{\mathfrak{q}}/tS_{\mathfrak{q}} = (S/tS)_{\mathfrak{q}/tS} \cong \bar{R}_{\bar{\mathfrak{q}}}$ is reduced. By Lemma 4.4, $S_{\mathfrak{q}}$ and hence its localization $(S_{\mathfrak{q}})_{\tilde{\mathfrak{p}}_{\mathfrak{q}}} = S_{\tilde{\mathfrak{p}}}$ is reduced. □

4.1.3. *Graded Cohen–Macaulay rings.* Let $(R, \mathfrak{m})$ be a Noetherian *local ring, that is, $\mathfrak{m}$ is the unique maximal graded ideal (see [BH93, Def. 1.5.13]). For any $\mathfrak{p} \in \operatorname{Spec} R$ denote by $\mathfrak{p}^*$ the maximal graded ideal contained in $\mathfrak{p}$. Then $\mathfrak{p}^* \in \operatorname{Spec} R$ (see [BH93, Lem. 1.5.6.(a)]) and (see [BH93, Thm. 1.5.8.(b)])

$$\mathfrak{p}^* \subsetneq \mathfrak{p} \implies \dim R_{\mathfrak{p}^*} + 1 = \dim R_{\mathfrak{p}} \tag{4.3}$$

If $\mathfrak{m} \in \operatorname{Max}\operatorname{Spec} R$ and $\mathfrak{p}^* \subsetneq \mathfrak{p}$, then $\mathfrak{p}^* \subsetneq \mathfrak{m}$ and hence $\dim R_{\mathfrak{p}^*} < \dim R_{\mathfrak{m}}$ and $\dim R_{\mathfrak{p}} \leqslant \dim R_{\mathfrak{m}}$ by (4.3). Otherwise, $\mathfrak{m} = \mathfrak{n}^* \subsetneq \mathfrak{n} \in \operatorname{Max}\operatorname{Spec} R$. Then $\dim R_{\mathfrak{p}^*} \leqslant \dim R_{\mathfrak{m}}$ and hence $\dim R_{\mathfrak{p}} \leqslant \dim R_{\mathfrak{m}} + 1 = \dim R_{\mathfrak{n}}$ by (4.3). It follows that

$$\dim R = \begin{cases} \dim R_{\mathfrak{m}} & \text{if } \mathfrak{m} \in \operatorname{Max}\operatorname{Spec} R, \\ \dim R_{\mathfrak{m}} + 1 & \text{if } \mathfrak{m} \notin \operatorname{Max}\operatorname{Spec} R. \end{cases} \tag{4.4}$$

For any proper graded ideal $I \lhd R$ also $(R/I, \mathfrak{m}/I)$ is *local and

$$\mathfrak{m} \in \operatorname{Max}\operatorname{Spec} R \iff \mathfrak{m}/I \in \operatorname{Max}\operatorname{Spec}(R/I). \tag{4.5}$$

All associated primes $\mathfrak{p} \in \operatorname{Ass} R$ are graded (see [BH93, Lem. 1.5.6.(b).(ii)]) and hence $\mathfrak{p} \subseteq \mathfrak{m}$. This yields a bijection (see [Sta18, Lemma 05BZ])

$$\operatorname{Ass} R \to \operatorname{Ass} R_{\mathfrak{m}}, \quad \mathfrak{p} \mapsto \mathfrak{p}_{\mathfrak{m}}. \tag{4.6}$$

Lemma 4.6. *Let $(R, \mathfrak{m})$ be a *local Cohen–Macaulay ring and $I \trianglelefteq R$ a graded ideal. Then R is pure-dimensional and* $\operatorname{height} I = \operatorname{codim} I$.

Proof. The hypothesis is equivalent to $R_{\mathfrak{m}}$ being (local) Cohen–Macaulay (see [BH93, Ex. 2.1.27.(c)]). In particular $R_{\mathfrak{m}}$ is pure-dimensional (see [BH93, Prop. 1.2.13]) and $\operatorname{height} I_{\mathfrak{m}} = \operatorname{codim} I_{\mathfrak{m}}$ (see [BH93, Cor. 2.1.4]). Using (4.4), (4.5) for $I = \mathfrak{p}$ and bijection (4.6),

$$\begin{aligned}
\forall \mathfrak{p} \in \operatorname{Ass} R\colon\ \dim R &= \begin{cases} \dim R_{\mathfrak{m}} + 1 & \text{if } \mathfrak{m} \in \operatorname{Max}\operatorname{Spec} R, \\ \dim R_{\mathfrak{m}} & \text{if } \mathfrak{m} \notin \operatorname{Max}\operatorname{Spec} R, \end{cases} \\
&= \begin{cases} \dim(R_{\mathfrak{m}}/\mathfrak{p}_{\mathfrak{m}}) + 1 & \text{if } \mathfrak{m} \in \operatorname{Max}\operatorname{Spec} R, \\ \dim(R_{\mathfrak{m}}/\mathfrak{p}_{\mathfrak{m}}) & \text{if } \mathfrak{m} \notin \operatorname{Max}\operatorname{Spec} R, \end{cases} \\
&= \begin{cases} \dim(R/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}} + 1 & \text{if } \mathfrak{m} \in \operatorname{Max}\operatorname{Spec} R, \\ \dim(R/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}} & \text{if } \mathfrak{m} \notin \operatorname{Max}\operatorname{Spec} R, \end{cases} \\
&= \dim(R/\mathfrak{p}).
\end{aligned}$$

Using (4.4) and (4.5),

$$\begin{aligned}
\operatorname{height} I &= \operatorname{height} I_{\mathfrak{m}} = \operatorname{codim} I_{\mathfrak{m}} \\
&= \dim R_{\mathfrak{m}} - \dim(R_{\mathfrak{m}}/I_{\mathfrak{m}}) \\
&= \dim R_{\mathfrak{m}} - \dim(R/I)_{\mathfrak{m}/I} \\
&= \dim R - \dim(R/I) = \operatorname{codim} I.
\end{aligned}$$

□

4.2. **Jacobian and degeneracy schemes.** In this subsection we associate Jacobian and second degeneracy schemes to a configuration. By results of Patterson and Kutz, their supports coincide and their codimension is at most 3.

If R is a Noetherian ring, then the minimal primes $\mathfrak{p} \in \operatorname{Min}\operatorname{Spec} R$ are the generic points of the associated affine scheme $\operatorname{Spec} R$. We refer to associated primes $\mathfrak{p} \in \operatorname{Ass} R$ as associated points of $\operatorname{Spec} R$. Due to Lemma 4.6,

$$\operatorname{codim}_{\mathbb{K}^E} \operatorname{Spec}(\mathbb{K}[x]/I) = \operatorname{height} I$$

for any graded ideal $I \trianglelefteq \mathbb{K}[x]$.

Definition 4.7 (Jacobian and degeneracy schemes)**.** Let $W \subseteq \mathbb{K}^E$ be a configuration. Then the subscheme

$$X_W := \operatorname{Spec}(\mathbb{K}[x]/\langle \psi_W \rangle) \subseteq \mathbb{K}^E$$

is called the *configuration hypersurface* of W. Its *Jacobian ideal* is

$$J_W := \langle \psi_W \rangle + \langle \partial_e \psi_W \mid e \in E \rangle \trianglelefteq \mathbb{K}[x].$$

The subschemes (see Definition 3.21)

$$\Sigma_W := \operatorname{Spec}(\mathbb{K}[x]/J_W) \subseteq \mathbb{K}^E, \quad \Delta_W := \operatorname{Spec}(\mathbb{K}[x]/M_W) \subseteq \mathbb{K}^E,$$

we call the *Jacobian scheme* and the *second degeneracy scheme* of W.

Remark 4.8 (Degeneracy and Non-smooth loci). If $\operatorname{ch}\mathbb{K} \nmid \operatorname{rk}\mathsf{M} = \deg\psi$ (see Remark 3.5), then ψ_W is a redundant generator of J_W due to the Euler identity. By Lemma 3.24, X_W^{red} and Δ_W^{red} are the first and second degeneracy loci of Q_W (see Definition 3.21) whereas Σ_W^{red} is the *non-smooth locus* of X_W over $\mathbb{K}$ (see [Mat89, Thm. 30.3.(1)]). If in addition $\mathbb{K}$ is perfect, then Σ_W^{red} is the *singular locus* of X_W (see [Mat89, §28, Lem. 1]). $\diamond$

Remark 4.9 (Codimension-2 components). The non-smooth locus Σ_W^{red} contains the intersection of any two irreducible components of X_W (see [Mat89, Thm. 30.3.(5)]). By Proposition 3.10, it follows that Σ_W has codimension 2 in $\mathbb{K}^E$ if M is disconnected even when loops are removed. $\diamond$

Lemma 4.10 (Inclusions of schemes). *For any configuration $W \subseteq \mathbb{K}^E$, there are inclusions of schemes $\Delta_W \subseteq \Sigma_W \subseteq X_W \subseteq \mathbb{K}^E$.*

Proof. By Lemma 3.24, $\psi_W \in M_W$ and hence the second inclusion. Let $e \in E$ and choose a basis of W as in the proof of Lemma 3.26. By Lemma 3.24 and the matrix representations for Q_W in (3.22) and (3.23), $\partial_e \psi_W \in M_W$. Thus, $J_W \subseteq M_W$ and the first inclusion follows. $\square$

Remark 4.11 (Schemes for matroids of small rank). Let $W \subseteq \mathbb{K}^E$ be a realization of a connected matroid M.

(a) Suppose that $\operatorname{rk}\mathsf{M} = 1$. Then W is generated by $(1, \dots, 1)$ after scaling E and $\mathsf{M} = \mathsf{U}_{1,n}$ is uniform where $n = |E|$. It follows that $\psi_W = \sum_{e \in E} x_e$ and $\Delta_W = \Sigma_W = \varnothing$.

(b) Suppose that $\operatorname{rk}\mathsf{M} = 2$. Then ψ_W is a quadratic form and J_W is a prime ideal generated by linear forms. It follows that both Δ_W and Σ_W are $\mathbb{K}$-linear subspaces of $\mathbb{K}^E$ and hence integral schemes.

$\diamond$

Example 4.12 (Schemes associated to a triangle). Let M be a matroid on $E \in \mathcal{C}_{\mathsf{M}}$ with $|E| = 3$ and hence $\operatorname{rk}\mathsf{M} = |E| - 1 = 2$. Up to scaling and ordering $E = \{e_1, e_2, e_3\}$ any realization W of M has the basis

$$w^1 := e_1 + e_3, \quad w^2 := e_2 + e_3.$$

With respect to this basis

$$Q_W = \begin{pmatrix} x_1 + x_3 & x_3 \\ x_3 & x_2 + x_3 \end{pmatrix}.$$

It follows $M_W = \langle x_1 + x_3, x_2 + x_3, x_3\rangle$ and Δ_W is a reduced $\mathbb{K}$-valued point.

On the other hand

$$\psi_W = \det(Q_W) = x_1x_2 + x_1x_3 + x_2x_3$$

and hence $J_W = \langle \psi_W, x_1 + x_2, x_1 + x_3, x_2 + x_3\rangle$. The matrix expressing the linear generators in terms of x_1, x_2, x_3 has determinant 2. It follows that Σ_W is reduced if and only if $\operatorname{ch}\mathbb{K} \neq 2$. $\diamond$

Patterson proved the following result (see [Pat10, Thm. 4.1]).

Theorem 4.13 (Patterson). *Let $W \subseteq \mathbb{K}^E$ be a configuration. Then there is an equality of reduced loci $\Sigma_W^{\mathrm{red}} = \Delta_W^{\mathrm{red}}$. In particular, Σ_W and Δ_W have the same generic points.* $\square$

Remark 4.14. While Patterson assumes $\operatorname{ch}\mathbb{K} = 0$ and excludes the generator $\psi_W \in J_W$, his proof works in general (see Remark 4.8). $\diamond$

Corollary 4.15 (Cremona automorphism of the torus). *Let $W \subseteq \mathbb{K}^E$ be a configuration. Then the automorphism of $\mathbb{T}^E$ defined by $x_e \mapsto y_e$ where $x_e \cdot y_e = 1$ for all $e \in E$ identifies*

$$X_W \cap \mathbb{T}^E \cong X_{W^\perp} \cap \mathbb{T}^E, \quad \Sigma_W \cap \mathbb{T}^E \cong \Sigma_{W^\perp} \cap \mathbb{T}^E, \quad \Delta_W \cap \mathbb{T}^E \cong \Delta_{W^\perp} \cap \mathbb{T}^E.$$

In particular, Σ_W, Δ_W, $\Sigma_{W^\perp}$, $\Delta_{W^\perp}$ have the same generic points in $\mathbb{T}^E$.

Proof. Propositions 3.12 and 3.25 yield the statements for X_W and Δ_W. Since $x_e\partial_{x_e} = y_e\partial_{y_e}$, the statement for Σ_W follows from that for X_W. The particular claim uses Theorem 4.13. $\square$

Proposition 4.16 (Codimension bound). *Let $W \subseteq \mathbb{K}^E$ be a configuration. Then the codimensions of Σ_W and Δ_W in $\mathbb{K}^E$ are bounded by*

$$\operatorname{codim}_{\mathbb{K}^E} \Sigma_W = \operatorname{codim}_{\mathbb{K}^E} \Delta_W \leqslant 3.$$

In case of equality, Δ_W is Cohen–Macaulay and hence pure-dimensional and Σ_W is equidimensional. Then all associated points of Δ_W are generic and all generic points of Σ_W have codimension 3 in $\mathbb{K}^E$.

Proof. The equality of codimensions follows from Theorem 4.13. The scheme Δ_W is defined by the ideal M_W of submaximal minors of the symmetric matrix Q_W with entries in $\mathbb{K}[x]$. Kutz proved the inequality and that M_W is a perfect ideal in case of equality (see [Kut74, Thm. 1]). In this latter case $\mathbb{K}[x]/M_W = \mathbb{K}[\Delta_W]$ is a Cohen–Macaulay ring (see [BH93, Thm. 2.1.5.(a), 2.1.9]). The remaining claims are due to Lemma 4.6 and Theorem 4.13. $\square$

4.3. **Deletion of (co)loops.** In this section we consider a matroid that is connected after deletion of all (co)loops. Here the Jacobian and second degeneracy schemes can be described explicitly. In addition to components of the connected deletion the (co)loops give rise to components of codimension 2.

Lemma 4.17. *Let R be a ring, $I \trianglelefteq R$ an ideal and $p \in I$. Then*

$$x \cdot I[x] + \langle p \rangle = I[x] \cap \langle p, x \rangle \trianglelefteq R[x],$$

where x is an indeterminate.

Proof. The non-trivial inclusion $\supseteq$ follows from $x \cdot I[x] = I[x] \cap \langle x \rangle$. □

Lemma 4.18 (Ideals and deletion of (co)loops)**.** *Let M be a matroid with realization W. For any $e \in E$*

$$J_W = \begin{cases} J_{W\setminus e}[x_e] & \text{if } e \text{ is a loop,} \\ J_{W\setminus e}[x_e] \cap \big\langle \psi_{W\setminus e}, x_e \big\rangle & \text{if } e \text{ is a coloop,} \end{cases}$$

and

$$M_W = \begin{cases} M_{W\setminus e}[x_e] & \text{if } e \text{ is a loop,} \\ M_{W\setminus e}[x_e] \cap \big\langle \psi_{W\setminus e}, x_e \big\rangle & \text{if } e \text{ is a coloop.} \end{cases}$$

Proof. By Proposition 3.14 and Lemma 3.26, the claim is clear if e is a loop. Suppose that e is a coloop. Note that $\psi_{W\setminus e} \in J_{W\setminus e}$ by definition and $\psi_{W\setminus e} \in M_{W\setminus e}$ by Lemma 3.24. By Proposition 3.14 and Lemma 4.17,

$$J_W = \big\langle \psi_{W\setminus e} \big\rangle + x_e \cdot J_{W\setminus e}[x_e] = J_{W\setminus e}[x_e] \cap \big\langle \psi_{W\setminus e}, x_e \big\rangle.$$

By Lemmas 3.26 and 4.17,

$$M_W = \big\langle \psi_{W\setminus e} \big\rangle + x_e \cdot M_{W\setminus e}[x_e] = M_{W\setminus e}[x_e] \cap \big\langle \psi_{W\setminus e}, x_e \big\rangle.$$ □

Proposition 4.19 (Schemes and deletion of (co)loops)**.** *Let M be a matroid with realization W. Denote by $L, C \subseteq E$ the sets of loops and coloops of M. Consider $\mathsf{M}' := \mathsf{M}\setminus(L \cup C)$ and $W' := W\setminus(L \cup C)$. Then*

$$\Sigma_W = \big(\Sigma_{W'} \times \mathbb{K}^{L\cup C}\big) \cup \bigcup_{e\in C} \big(X_{W'} \times \mathbb{K}^{L\cup C\setminus\{e\}}\big) \cup \bigcup_{C\ni e\neq f\in C} V(x_e, x_f)$$

and

$$\Delta_W = \big(\Delta_{W'} \times \mathbb{K}^{L\cup C}\big) \cup \bigcup_{e\in C} \big(X_{W'} \times \mathbb{K}^{L\cup C\setminus\{e\}}\big) \cup \bigcup_{C\ni e\neq f\in C} V(x_e, x_f).$$

Proof. By Lemma 4.18 and induction on $|L \cup C|$, we have

$$J_W = J_{W'}[x_{L\cup C}] \cap \bigcap_{e\in C} \langle \psi_{W'}, x_e \rangle \cap \bigcap_{C\ni e\neq f\in C} \langle x_e, x_f \rangle$$

and

$$M_W = M_{W'}[x_{L\cup C}] \cap \bigcap_{e\in C} \langle \psi_{W'}, x_e \rangle \cap \bigcap_{C\ni e\neq f\in C} \langle x_e, x_f \rangle.$$

The claim follows by taking associated affine schemes. □

4.4. **Generic points and codimension.** In this subsection we show that the Jacobian and second degeneracy schemes reach the codimension bound of 3 in case of connected matroids. The statements on codimension and Cohen–Macaulayness in our main result follow. In the process we obtain a description of the generic points in relation with any non-disconnective handle.

Lemma 4.20 (Primes over the Jacobian ideal)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a connected matroid M, and let $H \in \mathcal{H}_{\mathsf{M}}$ be a proper handle.*

(a) For any $h \in H$, $x^{H\setminus\{h\}} \cdot \psi_{W\setminus H} \in J_W$.
(b) For any $e, f \in H$ with $e \neq f$, $x^{H\setminus\{e,f\}} \cdot \psi_{W\setminus H} \in J_W + \langle x_e, x_f\rangle$.
(c) For any $d \in H$ and $e \in E\setminus H$, $x^{H\setminus\{d\}} \cdot \partial_e\psi_{W\setminus H} \in J_W + \langle x_d\rangle$.
(d) If $\mathfrak{p} \in \operatorname{Spec}\mathbb{K}[x]$ with $J_W \subseteq \mathfrak{p} \not\ni \psi_{W\setminus H}$, then $\langle x_e, x_f, x_g\rangle \subseteq \mathfrak{p}$ for some $e, f, g \in H$ with $e \neq f \neq g \neq e$.

Proof. By Corollary 3.15, we may assume that

$$\psi_W = \sum_{h\in H} x^{H\setminus\{h\}} \cdot \psi_{W\setminus H} + x^H \cdot \psi_{W/H}.$$

(a) Using that ψ_W is a linear combination of square-free monomials (see Definition 3.2,

$$x^{H\setminus\{h\}} \cdot \psi_{W\setminus H} = \psi_W|_{x_h=0} = \psi_W - x_h \cdot \partial_h\psi_W \in J_W.$$

(b) This follows from

$$\begin{aligned} J_W \ni \partial_e\psi_W &= \sum_{h\in H} x^{H\setminus\{e,h\}} \cdot \psi_{W\setminus H} + x^{H\setminus\{e\}} \cdot \psi_{W/H} \\ &\equiv x^{H\setminus\{e,f\}} \cdot \psi_{W\setminus H} \mod \langle x_e, x_f\rangle. \end{aligned}$$

(c) This follows from

$$\begin{aligned} J_W \ni \partial_e\psi_W &= \sum_{h\in H} x^{H\setminus\{h\}} \cdot \partial_e\psi_{W\setminus H} + x^H \cdot \partial_e\psi_{W/H} \\ &\equiv x^{H\setminus\{d\}} \cdot \partial_e\psi_{W\setminus H} \mod \langle x_d\rangle. \end{aligned}$$

(d) By (a), the hypotheses force $x^{H\setminus\{h\}} \in \mathfrak{p}$ for all $h \in H$ and hence $\langle x_e, x_f\rangle \subseteq \mathfrak{p}$ for some $e, f \in H$ with $e \neq f$. Then $x^{H\setminus\{e,f\}} \in \mathfrak{p}$ by (b) and the claim follows.

□

Lemma 4.21 (Inductive codimension bound)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a connected matroid M, and let $H \in \mathcal{H}_{\mathsf{M}}$ be a proper non-disconnective handle. If $\operatorname{codim}_{\mathbb{K}^{E\setminus H}} \Sigma_{W\setminus H} = 3$, then Σ_W is equidimensional of codimension 3 in $\mathbb{K}^E$ with generic points of the following types:*

(a) $\mathfrak{p} = \langle x_e, x_f, x_g\rangle =: \mathfrak{p}_1$ for some $e, f, g \in H$, $e \neq f \neq g \neq e$,
(b) $\mathfrak{p} = \left\langle \psi_{W\setminus H}, x_d, x_e\right\rangle =: \mathfrak{p}_2$ for some $d, e \in H$, $d \neq e$,
(c) $\psi_{W\setminus H}, \psi_{W/H} \in \mathfrak{p} \not\ni x_e$, for all $e \in H$.

Proof. Since H is non-disconnective $\psi_{W\setminus H}$ is irreducible by Proposition 3.10. In particular, $\mathfrak{p}_i \in \operatorname{Spec}\mathbb{K}[x]$ with $\operatorname{height}\mathfrak{p}_i = 3$ for $i = 1, 2$.

Let $\mathfrak{p} \in \operatorname{Spec}\mathbb{K}[x]$ be any minimal prime over J_W. By Proposition 4.16, it suffices to show for the equidimensionality that $\operatorname{height}\mathfrak{p} \geqslant 3$. This follows in particular if $\mathfrak{p}$ contains a prime of type $\mathfrak{p}_1$ or $\mathfrak{p}_2$. By Lemma 4.20.(d), the former is the case if $\psi_{W\setminus H} \notin \mathfrak{p}$. We may thus assume that $\psi_{W\setminus H} \in \mathfrak{p}$.

First suppose that $x_d \in \mathfrak{p}$ for some $d \in H$. By Lemma 4.20.(c), then

$$x^{H\setminus\{d\}} \cdot \partial_e \psi_{W\setminus H} \in \mathfrak{p}$$

for all $e \in E\setminus H$. If $x^{H\setminus\{d\}} \in \mathfrak{p}$, then $\mathfrak{p}$ contains a prime of type $\mathfrak{p}_2$. Otherwise $J_{W\setminus H} + \langle x_d \rangle \subseteq \mathfrak{p}$. Since $J_{W\setminus H} \trianglelefteq \mathbb{K}[x_{E\setminus H}]$ but $d \in H$, the codimension hypothesis implies that

$$\operatorname{height}(J_{W\setminus H} + \langle x_d \rangle) = 4.$$

It follows that $\operatorname{height}\mathfrak{p} \geqslant 4$ which can not occur.

Now suppose that $x_h \notin \mathfrak{p}$ for all $h \in H$ and hence $\psi_{W/H} \in \mathfrak{p}$ by Corollary 3.15. By Lemma 4.20.(c), then

$$x^{H\setminus\{d\}} \cdot \partial_e \psi_{W\setminus H} \in \mathfrak{p} + \langle x_d \rangle$$

for any $d \in H$ and $e \in E\setminus H$. Thus any minimal prime $\mathfrak{q}$ over $\mathfrak{p} + \langle x_d \rangle$ contains one of the ideals

$$\langle \psi_{W\setminus H}, \psi_{W/H}, x_d, x_h \rangle, \quad J_{W\setminus H} + \langle x_d \rangle$$

for some $h \in H\setminus\{d\}$. Both have height at least 4: the first one since $\deg \psi_{W/H} < \deg \psi_{W\setminus H}$ by Lemma 2.3.(e) (see Remark 3.5) and $\psi_{W\setminus H}$ is irreducible, the second by hypothesis. Thus $\operatorname{height}\mathfrak{q} \geqslant 4$ and hence $\operatorname{height}(\mathfrak{p} + \langle x_d \rangle) \geqslant 4$ and then $\operatorname{height}\mathfrak{p} \geqslant 3$ by Lemma 4.1.(b). $\square$

Lemma 4.22 (Generic points for circuits)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M on $E \in \mathcal{C}_{\mathsf{M}}$ with $|E| - 1 = \operatorname{rk}\mathsf{M} \geqslant 2$. Then Σ_W^{red} is the union of all codimension-3 coordinate subspaces of $\mathbb{K}^E$.*

Proof. We apply the strategy of the proof of Lemma 4.21. Let $\mathfrak{p} \in \operatorname{Spec}\mathbb{K}[x]$ be any minimal prime over J_W. If $\psi_{W\setminus H} \notin \mathfrak{p}$ for some $E \neq H \in \mathcal{H}_{\mathsf{M}}$, then by Lemma 4.20.(d) $\mathfrak{p}$ contains x_e, x_f, x_g where $e, f, g \in H$ with $e \neq f \neq g \neq e$. Otherwise $\mathfrak{p}$ contains $x^{E\setminus H} = \psi_{W\setminus H} \in \mathfrak{p}$ for all $E \neq H \in \mathcal{H}_{\mathsf{M}}$ and hence all x_e where $e \in E$ (which can only occur if $|E| = 3$). By Proposition 4.16, it follows that $\mathfrak{p} = \langle x_e, x_f, x_g \rangle$ where $e \neq f \neq g \neq e$. By symmetry, all such triples $e, f, g \in E$ occur (see Example 3.7). $\square$

Theorem 4.23 (Cohen–Macaulayness of degeneracy schemes)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a connected matroid M of rank $\operatorname{rk}\mathsf{M} \geqslant 2$. Then Δ_W is Cohen–Macaulay (and hence pure-dimensional) and Σ_W is equidimensional of codimension 3.*

Proof. By Proposition 4.16, it suffices to show that $\operatorname{codim}_{\mathbb{K}^E} \Sigma_W = 3$. Lemma 2.9 yields a circuit $C \in \mathcal{C}_{\mathsf{M}}$ of size $|C| \geqslant 3$ and $\operatorname{codim}_{\mathbb{K}^C} \Sigma_{W|C} = 3$ by Lemma 4.22. By Lemma 4.21 and induction over a handle decomposition as in Proposition 2.5, then also $\operatorname{codim}_{\mathbb{K}^E} \Sigma_W = 3$. □

Corollary 4.24 (Types of generic points)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a connected matroid M of rank $\operatorname{rk} \mathsf{M} \geqslant 2$, and let $H \in \mathcal{H}_{\mathsf{M}}$ be a non-disconnective handle such that $\operatorname{rk}(\mathsf{M} \backslash H) \geqslant 2$. Then each generic point $\mathfrak{p}$ of Σ_W is of a type listed in Lemma 4.21 with respect to H.*

Proof. Applying Theorem 4.23 to the matroid $\mathsf{M} \backslash H$ with realization $W \backslash H$ the claim follows from Lemma 4.21. □

Remark 4.25. In the presence of a disconnective handle $H \in \mathcal{H}_{\mathsf{M}}$ other types of generic points of Σ_W may appear such as $\langle \psi_1, \psi_2, x_d \rangle$ where $\psi_{W \backslash H} = \psi_1 \cdot \psi_2$ and $d \in H$. ◇

Corollary 4.26 (Generic points for 3-connected matroids)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a 3-connected matroid M of rank $\operatorname{rk} \mathsf{M} \geqslant 2$ with $|E| > 3$. Then all generic points of Σ_W lie in $\mathbb{T}^E$.*

Proof. Let $\mathfrak{p}$ be a generic point of Σ_W and pick any $e \in E$. By Proposition 2.4, $H := \{e\} \in \mathcal{H}_{\mathsf{M}}$ is a non-disconnective 1-handle. Moreover $\operatorname{rk} \mathsf{M} \backslash H = \operatorname{rk} \mathsf{M} \geqslant 2$ by Lemma 2.3.(e). Then Corollary 4.24 forces $\mathfrak{p}$ to be of type (c) in Lemma 4.21, that is, $\mathfrak{p} \notin V(x_e)$. It follows that $\mathfrak{p} \in \bigcap_{e \in E} D(x_e) = \mathbb{T}^E$. □

4.5. **Reducedness of degeneracy schemes.** In this subsection we prove the reducedness statement in our main result following the strategy outlined in the introduction.

Lemma 4.27 (Reducedness for the prism)**.** *Let W be any realization of the prism matroid M (see Example 2.18). Then $\Delta_W \cap D(x_1 \cdots x_6)$ is a reduced linear variety of codimension 3, defined by 3 linear binomials each supported in one of the handles. If $\operatorname{ch} \mathbb{K} \neq 2$, then also $\Sigma_W \cap D(x_1 \cdots x_6)$ is reduced.*

Proof. By Lemma 2.19, the matrix of Q_W can be chosen to be (see Definition 3.21)

$$Q_W = \begin{pmatrix} x_1 + x_2 & 0 & 0 & x_1 \\ 0 & x_3 + x_4 & 0 & x_3 \\ 0 & 0 & x_5 + x_6 & x_5 \\ x_1 & x_3 & x_5 & x_1 + x_3 + x_5 \end{pmatrix}.$$

Reducing its entries modulo $\mathfrak{p} := \langle x_1 + x_2, x_3 + x_4, x_5 + x_6\rangle$ makes all its 3×3-minors 0. Therefore $J_W \subseteq M_W \subseteq \mathfrak{p}$. Using the minors

$$\begin{aligned} Q_W(3,2) &= (x_1 + x_2) \cdot (-x_3 x_5), \\ Q_W(4,2) &= (x_1 + x_2) \cdot (-x_3) \cdot (x_5 + x_6), \\ Q_W(4,3) &= (x_1 + x_2) \cdot (x_3 + x_4) \cdot x_5, \\ Q_W(4,4) &= (x_1 + x_2) \cdot (x_3 + x_4) \cdot (x_5 + x_6), \end{aligned}$$

one computes that

$$Q_W(4,4) - Q_W(4,3) + Q_W(4,2) - Q_W(3,2) = (x_1 + x_2) \cdot x_4 x_6.$$

By symmetry, it follows that $x_2 x_4 x_6 \cdot \mathfrak{p} \subseteq M_W$ and hence

$$\Delta_W \cap D(x_2 x_4 x_6) = V(\mathfrak{p}) \cap D(x_2 x_4 x_6).$$

With $\psi_W = \det(Q_W)$ one computes that

$$\begin{aligned} (x_2 \cdot (x_2 \partial_2 - 1) + x_4 x_6 \cdot (\partial_3 + \partial_5) + (x_4 + x_6) \cdot (1 - x_4 \partial_4 - x_6 \partial_6))\psi_W \\ = 2 \cdot (x_1 + x_2) \cdot x_4^2 x_6^2. \end{aligned}$$

By symmetry, it follows that $2 \cdot x_2^2 x_4^2 x_6^2 \cdot \mathfrak{p} \subseteq J_W$ and hence

$$\Sigma_W \cap D(x_2 x_4 x_6) = V(\mathfrak{p}) \cap D(x_2 x_4 x_6).$$

if $\operatorname{ch} \mathbb{K} \neq 2$. ☐

More details on the prism matroid can be found in Example 5.1.

Lemma 4.28 (Reduction and deleting non-(co)loops)**.** *Let $e \in E$ be a non-(co)loop of a matroid M. Identify $\mathbb{K}[x]/\langle x_e\rangle = \mathbb{K}[x_{E\setminus\{e\}}]$ and set $\bar{I} := (I + \langle x_e\rangle)/\langle x_e\rangle \trianglelefteq \mathbb{K}[x]/\langle x_e\rangle$ for any $I \trianglelefteq \mathbb{K}[x]$. Then $J_{W\setminus e} \subseteq \bar{J}_W$ and $M_{W\setminus e} = \bar{M}_W$ for any realization W of M.*

Proof. This follows from Proposition 3.14 and Lemma 3.26. ☐

Lemma 4.29 (R_0 and deleting non-(co)loops)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M, and let $e \in E$ be a non-(co)loop. Then $\Sigma_{W\setminus e} = \varnothing$ implies $\Sigma_W = \varnothing$. Suppose that $D(x_e)$ contains all generic points of Σ_W and that Σ_W and $\Sigma_{W\setminus e}$ are equidimensional of the same codimension. If $\Sigma_{W\setminus e}$ is R_0, then Σ_W is R_0. In this case, each $\mathfrak{p} \in \operatorname{Min}\Sigma_W$ defines a subset $\varphi(\mathfrak{p}) \subseteq \operatorname{Min}\Sigma_{W\setminus e}$ such that*

$$V(\mathfrak{p}) \cap V(x_e) = \bigcup_{\mathfrak{q}\in\varphi(\mathfrak{p})} V(\mathfrak{q})$$

and $\varphi(\mathfrak{p}) \cap \varphi(\mathfrak{p}') = \varnothing$ for $\mathfrak{p} \neq \mathfrak{p}'$. In particular,

$$|\operatorname{Min}\Sigma_W| \leqslant \left|\operatorname{Min}\Sigma_{W\setminus e}\right|.$$

The same statements hold for Σ replaced by Δ.

Proof. With notation from Lemma 4.28 the subscheme $\Sigma_W \cap V(x_e) \subseteq \mathbb{K}^{E\setminus\{e\}}$ is defined by the ideal $\bar{J}_W$. By Lemma 4.28 and homogeneity,

$$\begin{aligned}\Sigma_{W\setminus e} = \varnothing \iff J_{W\setminus e} = \mathbb{K}[x_{E\setminus\{e\}}] &\implies J_W + \langle x_e\rangle = \mathbb{K}[x]\\ &\implies J_W = \mathbb{K}[x] \iff \Sigma_W = \varnothing\end{aligned}$$

which is the first claim.

Any generic point of Σ_W is represented by a prime $\mathfrak{p} \in \operatorname{Spec}\mathbb{K}[x]$ minimal over J_W. Let $\mathfrak{q} \in \operatorname{Spec}\mathbb{K}[x]$ be minimal over $\mathfrak{p} + \langle x_e\rangle$ and set $\bar{\mathfrak{q}} := \mathfrak{q}/\langle x_e\rangle \in \operatorname{Spec}\mathbb{K}[x_{E\setminus\{e\}}]$. Since $x_e \notin \mathfrak{p}$ Lemma 4.1 shows that

$$\operatorname{height}\mathfrak{q} = \operatorname{height}\mathfrak{p} + 1, \quad \operatorname{height}\bar{\mathfrak{q}} = \operatorname{height}\mathfrak{q} - \operatorname{height}\langle x_e\rangle = \operatorname{height}\mathfrak{p}.$$

By Lemmas 4.6 and 4.28 and the dimension hypothesis, it follows that $\bar{\mathfrak{q}}$ is minimal over both $\bar{J}_W$ and $J_{W\setminus e}$ and hence represents a generic point of both $\Sigma_W \cap V(x_e)$ and $\Sigma_{W\setminus e}$.

Consider now $\mathfrak{p}$ and $\mathfrak{q}$ as elements of Σ_W and denote by $t \in \mathbb{K}[\Sigma_W]$ the image of x_e. Then $\mathfrak{q}$ is minimal over t and hence t a parameter of $R := \mathbb{K}[\Sigma_W]_{\mathfrak{q}}$. By Lemma 4.4, R is a domain with unique minimal prime $\mathfrak{p}_{\mathfrak{q}}$. Thus $\mathbb{K}[\Sigma_W]_{\mathfrak{p}} = R_{\mathfrak{p}}$ is reduced and $\mathfrak{p}$ is uniquely determined by $\mathfrak{q}$. With $\varphi(\mathfrak{p})$ the set of all possible $\mathfrak{q}$ the remaining claims follow.

The preceding arguments remain valid if Σ and J are replaced by Δ and M respectively. ☐

Lemma 4.30 (Initial terms and contracting non-(co)loops)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M. Suppose $E = F \sqcup G$ is partitioned in such a way that M/G is obtained by successively contracting non-(co)loops. For any ideal $J \trianglelefteq \mathbb{K}[x]_{x^G}$ denote by $J^{\inf}$ the ideal generated by the lowest x_F-degree parts of the elements of J. Then $J_{W/G}[x_G^{\pm 1}] \subseteq (J_W)^{\inf}_{x^G}$ and $M_{W/G}[x_G^{\pm 1}] \subseteq (M_W)^{\inf}_{x^G}$.*

Proof. We iterate Proposition 3.14 and Lemma 3.26 respectively to pass from W to W/G by successively contracting non-(co)loops $e \in G$. This yields a basis of W extending a basis $w^1, \dots, w^s$ of W/G such that, for all $i, j \in 1, \dots, s$,

$$\psi_W = x^G \cdot \psi_{W/G} + p, \quad Q_W(i,j) = x^G \cdot Q_{W/G}(i,j) + q_{i,j},$$

where $p, q_{i,j} \in \mathbb{K}[x]$ are polynomials with no term divisible by x^G. Both ψ_W and $Q_W(i,j)$ are homogeneous $\mathbb{K}$-linear combinations of square-free monomials (see Definition 3.2 and Lemma 3.26). It follows that $x^G \cdot \psi_{W/G}$ and $x^G \cdot Q_{W/G}(i,j)$ are the respective lowest x_F-degree parts of ψ_W and $Q_W(i,j)$. The claimed inclusions follow. ☐

Lemma 4.31 (R_0 and contracting non-(co)loops)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M. Suppose $E = F \sqcup G$ is partitioned in such a way that M/G is obtained by successively contracting non-(co)loops. Then $\Sigma_{W/G} = \varnothing$ implies $\Sigma_W \cap D(x^G) \cap V(x_F) = \varnothing$. Suppose that $\Sigma_W \cap D(x^G)$ and $\Sigma_{W/G}$ are equidimensional of the same codimension. If*

$\Sigma_{W/G}$ is R_0, then $\Sigma_W \cap D(x^G)$ is R_0 along $V(x_F)$. The same statements hold for Σ replaced by Δ.

Proof. Consider the ideal

$$\begin{aligned} I := \langle x_F \rangle \trianglelefteq \mathbb{K}[\Sigma_W \cap D(x^G)] &=: R \\ &= \mathbb{K}[\Sigma_W]_{x^G} = (\mathbb{K}[x_E]/J_W)_{x^G} = \mathbb{K}[x_F, x_G^{\pm 1}]/(J_W)_{x^G} \end{aligned}$$

where R is equidimensional by hypothesis. With notation from Lemma 4.30

$$\bar{R} = \operatorname{gr}_I R = \operatorname{gr}_I(\mathbb{K}[x_F, x_G^{\pm 1}]/(J_W)_{x^G}) = \mathbb{K}[x_F, x_G^{\pm 1}]/(J_W)_{x^G}^{\inf}.$$

Lemma 4.30 then yields the first claim

$$\begin{aligned} \Sigma_{W/G} = \varnothing &\iff J_{W/G} = \mathbb{K}[x_F] \implies \bar{R} = 0 \\ &\iff I = R \iff \Sigma_W \cap D(x^G) \cap V(x_F) = \varnothing. \end{aligned}$$

We may assume now that $I \neq R$, as otherwise $\Sigma_W \cap D(x^G) \cap V(x_F) = \varnothing$ makes the second claim void. By Lemma 4.5.(a) and the equidimensionality hypotheses, the rings $\bar{R}$ and

$$\mathbb{K}[x_F, x_G^{\pm 1}]/(J_{W/G}[x_G^{\pm 1}]) = (\mathbb{K}[x_F]/J_{W/G})[x_G^{\pm 1}] = \mathbb{K}[\Sigma_{W/G} \times \mathbb{T}^G]$$

are equidimensional of the same dimension. By Lemma 4.30, the former is a homomorphic image of the latter. It follows that

$$\operatorname{Min}\operatorname{Spec} \bar{R} \subseteq \operatorname{Min}(\Sigma_{W/G} \times \mathbb{T}^G).$$

Hence, if $\Sigma_{W/G}$ is R_0 then so is $\bar{R}$. By Lemma 4.5.(b), then R is R_0 along $V(I)$. This means that $\Sigma_W \cap D(x^G)$ is R_0 along $V(x_F)$.

The preceding arguments remain valid if Σ and J are replaced by Δ and M respectively. $\square$

Lemma 4.32 (R_0 for circuits)**.** *Let W be a realization of a matroid M on $E \in \mathcal{C}_\mathsf{M}$ of rank $\operatorname{rk}\mathsf{M} = |E| - 1 \geqslant 2$. Then Δ_W is R_0. If $\operatorname{ch}\mathbb{K} \neq 2$, then also Σ_W is R_0.*

Proof. We proceed by induction over $|E|$. The case $|E| = 3$ is covered by Example 4.12. Suppose now that $|E| > 3$.

By Lemma 4.22, each generic point of Σ_W is of the form $\mathfrak{p} = \langle x_e, x_f, x_g \rangle$ for some $e, f, g \in H$ with $e \neq f \neq g \neq e$. Pick $d \in E \setminus \{e, f, g\}$. Then $E \setminus \{d\} \in \mathcal{C}_{\mathsf{M}/d}$ and hence $\Sigma_{W/d}$ is R_0 by induction. By Lemmas 4.2 and 4.31, $\Sigma_W \cap D(x_d)$ is then R_0 along $V(x_{E\setminus\{d\}})$. But $\mathfrak{p} \in D(x_d)$ and $V(x_{E\setminus\{d\}}) \subseteq V(\mathfrak{p})$ by choice of d. This means that Σ_W is reduced at $\mathfrak{p}$ and hence R_0.

By Theorem 4.13, Δ_W has the same generic points as Σ_W. Therefore the preceding arguments remain valid if Σ is replaced by Δ. $\square$

Lemma 4.33 (R_0 and contraction of non-maximal handles)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a connected matroid M of rank $\operatorname{rk}\mathsf{M} \geqslant 2$. Assume that $|\operatorname{Max}\mathcal{H}_\mathsf{M}| \geqslant 2$ and set*

$$\hbar := |E| - |\operatorname{Max}\mathcal{H}_\mathsf{M}| \geqslant 0.$$

Suppose that $\Sigma_{W'}$ is R_0 for every realization $W' \subseteq \mathbb{K}^{E'}$ of every connected matroid M' of rank $\operatorname{rk}\mathsf{M}' \geqslant 2$ with $|E'| < |E|$.

(a) If $\hbar > 3$, then Σ_W is R_0.

(b) If only $\hbar > 2$, then Σ_W is reduced at all generic points $\mathfrak{p}$ with $\mathfrak{p} \in V(x_e)$ for some $e \in E$.

The same statements hold for Σ replaced by Δ.

Proof. Let $\mathfrak{p} \in \operatorname{Spec}\mathbb{K}[x]$ with $\operatorname{height}\mathfrak{p} = 3$. Pick a subset $F \subseteq E$ such that $|F \cap H'| = 1$ for all $H' \in \operatorname{Max}\mathcal{H}_M$. In this process, if $x_e \in \mathfrak{p}$ and $e \in H' \in \operatorname{Max}\mathcal{H}_M$, then take $F \cap H' = \{e\}$. If $\hbar > 3$, then by Lemma 4.1.(b)

$$\operatorname{height}(\mathfrak{p}+\langle x_F\rangle) \leqslant 3+|F| = 3+|\operatorname{Max}\mathcal{H}_{\mathsf{M}}| < |E| = \operatorname{height}\langle x_E\rangle. \tag{4.7}$$

If $\hbar > 2$ and $x_e \in \mathfrak{p}$, then (4.7) holds with 3 replaced by 2. Pick $\mathfrak{q} \in \operatorname{Spec}\mathbb{K}[x]$ such that

$$\mathfrak{p} + \langle x_F\rangle \subseteq \mathfrak{q} \subsetneq \langle x_E\rangle. \tag{4.8}$$

Add to F all $f \in E$ with $x_f \in \mathfrak{q}$. This does not affect (4.8). Then $x_g \notin \mathfrak{q}$ and hence $x_g \notin \mathfrak{p}$ for all $g \in G := E\backslash F \neq \varnothing$. In other words,

$$\mathfrak{p} \in D(x^G), \quad \mathfrak{q} \in V(\mathfrak{p}) \cap D(x^G) \cap V(x_F) \neq \varnothing. \tag{4.9}$$

By the initial choice of F, we see that $G \cap H' \subsetneq H'$ for each $H' \in \operatorname{Max}\mathcal{H}_{\mathsf{M}}$. By Lemma 2.3.(d), successively contracting all elements of G does not affect circuits and maximal handles, up to bijection, and therefore preserves connectedness. In particular M/G is a connected matroid on the set F and obtained by successively contracting non-(co)loops.

Since $|F| \geqslant |\operatorname{Max}\mathcal{H}_{\mathsf{M}}| \geqslant 2$ connectedness implies $\operatorname{rk}(\mathsf{M}/G) \geqslant 1$. If $\operatorname{rk}(\mathsf{M}/G) = 1$, then $\Sigma_{W/G} = \varnothing$ by Remark 4.11.(a). Then $\Sigma_W \cap D(x^G) \cap V(x_F) = \varnothing$ by Lemma 4.31 and hence $\mathfrak{p} \notin \Sigma_W$ by (4.9).

Suppose now $\mathfrak{p} \in \Sigma_W$ and hence $\operatorname{rk}(\mathsf{M}/G) \geqslant 2$. Then $\Sigma_{W/G}$ is R_0 by hypothesis, and $\mathfrak{p} \in \Sigma_W \cap D(x^G)$ is along $V(x_F)$ by (4.9). By Theorem 4.23 and Lemma 4.2, $\Sigma_W \cap D(x^g)$ and $\Sigma_{W/G}$ are equidimensional of codimension 3. By Lemma 4.31, Σ_W is thus reduced at $\mathfrak{p}$ and the claims follow.

The preceding arguments remain valid if Σ is replaced by Δ. $\square$

Theorem 4.34 (Reducedness of degeneracy schemes)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a connected matroid M of rank $\operatorname{rk}\mathsf{M} \geqslant 2$. Then Δ_W is reduced and agrees with Σ_W^{red}. If $\operatorname{ch}\mathbb{K} \neq 2$, then Σ_W is generically reduced.*

Proof. By Theorem 4.23, Δ_W is pure-dimensional. By Theorem 4.13, the first claim follows if Σ_W is R_0.

First assume that $\operatorname{ch}\mathbb{K} \neq 2$. We proceed by induction over $|E|$. By Lemma 4.32, Σ_W is R_0 if $E \in \mathcal{C}_{\mathsf{M}}$. Otherwise, by Proposition 2.5, M

has a handle decomposition of length $k \geqslant 2$. By Proposition 2.8, M has

$$\ell \geqslant k+1 \geqslant 3 \tag{4.10}$$

(disjoint) non-disconnective handles $H = H_1, \dots, H_\ell \in \mathcal{H}_{\mathsf{M}}$. Note that $H_1, \dots, H_\ell \in \operatorname{Max} \mathcal{H}_{\mathsf{M}} \cap \mathcal{I}_{\mathsf{M}}$ by Lemma 2.3.(c) and (b). In particular $\operatorname{rk}(\mathsf{M}\backslash H) \neq 0$.

Suppose first that $H = \{h\}$. Then $\operatorname{rk}(\mathsf{M}\backslash h) \geqslant 2$ by Lemma 4.29 and by Theorem 4.23 both Σ_W and $\Sigma_{W\backslash h}$ are equidimensional of codimension 3. By Corollary 4.24, we have $x_h \notin \mathfrak{p}$ for all generic points $\mathfrak{p}$ of Σ_W. Thus Σ_W is R_0 by Lemma 4.29 and the induction hypothesis.

Suppose now that $|H_i| \geqslant 2$ for all $i = 1, \dots, \ell$. If $\hbar := |E| - |\operatorname{Max} \mathcal{H}_{\mathsf{M}}| > 3$, then Σ_W is R_0 by Lemma 4.33 and the induction hypothesis. Otherwise with $m := |\operatorname{Max} \mathcal{H}_{\mathsf{M}}|$

$$2\ell + (m-\ell) \leqslant \sum_{i=1}^{\ell} |H_i| + (m-\ell) \leqslant |E| = \hbar + m \leqslant 3+m$$

and hence $2\ell \leqslant \sum_{i=1}^{\ell} |H_i| \leqslant 3+\ell$. Comparing with (4.10) we must have $\ell = 3$ and $k = 2$ and $|H_i| = 2$ for $i = 1,2,3$. By Lemma 2.7, $E = H_1 \sqcup H_2 \sqcup H_3$ is then the handle partition. In particular $\hbar = 6-3 = 3 > 2$. By Lemma 2.19, M is the prism matroid. Then Σ_W is reduced at generic points of type 4.21.(b) by Lemma 4.33 and the induction hypothesis and of type 4.21.(c) by Lemma 4.27. There are no generic points of type 4.21.(a) since $|H_i| < 3$ for $i = 1,2,3$. By Corollary 4.24, there are no other types of generic points.

The preceding arguments are valid for arbitrary $\operatorname{ch} \mathbb{K}$ if Σ is replaced by Δ. □

Corollary 4.35 (Reduced degeneracy scheme with (co)loops)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a matroid M. Suppose that M is connected after deletion of all (co)loops. Then Δ_W is reduced.*

Proof. This follows from Propositions 3.10 and 4.19, Remark 4.11 and Theorem 4.34. □

Corollary 4.35 gives evidence for the following conjecture.

Conjecture 4.36 (Reduced degeneracy scheme)**.** For any configuration $W \subseteq \mathbb{K}^E$, the scheme Δ_W is reduced.

4.6. **Irreducibility of Jacobian schemes.** In this subsection we prove the following companion result to Proposition 3.10.

Theorem 4.37 (Irreducibility of Jacobian schemes)**.** *Let $W \subseteq \mathbb{K}^E$ be a realization of a 3-connected matroid M of rank $\operatorname{rk}_{\mathsf{M}} \geqslant 2$. Then the scheme Σ_W is irreducible and the scheme Δ_W is integral.*

Proof. By Remark 4.11.(b), the claim holds if $\operatorname{rk} \mathsf{M} = 2$. If $|E| \leqslant 4$, then $\mathsf{M} = \mathsf{U}_{2,n}$ where $n \in \{3,4\}$ (see [Oxl11, Tab. 8.1]) and $\operatorname{rk} \mathsf{M} = 2$. We may thus assume that $\mathrm{rk}_{\mathsf{M}} \geqslant 3$ and $|E| \geqslant 5$. By Theorem 4.34, the claim on Σ_W implies that on Δ_W. The former follows from Lemmas 4.38, 4.42, 4.43 and Corollary 4.41 below. ☐

In the following we use notation from Example 2.20.

Lemma 4.38 (Reduction to wheels and whirls)**.** *It suffices to verify Theorem 4.37 for $\mathsf{M} \in \{\mathsf{W}_n, \mathsf{W}^n\}$ with $n \geqslant 3$.*

Proof. Let M and W be as in Theorem 4.37. Since 3-connectedness is invariant under duality also $\mathsf{M}^\perp$ satisfies the hypotheses on M. By Corollary 4.26, the generic points of both Σ_W and $\Sigma_{W^\perp}$ lie in $\mathbb{T}^E$. By Corollary 4.15, irreducibility is thus equivalent for Σ_W and $\Sigma_{W^\perp}$.

We proceed by induction on $|E|$. The base case $|E| \leqslant 4$ is covered. Suppose that M is not a wheel or a whirl. Since $\mathrm{rk}\,\mathsf{M} \geqslant 3$, Tutte's Wheels and Whirls Theorem (see [Oxl11, p. 8.8.4]) yields an $e \in E$ such that $\mathsf{M}\backslash e$ or M/e is again 3-connected. We may assume the latter case (see (2.6)). The scheme $\Sigma_{W/e}$ is then irreducible by induction hypothesis. By Lemma 4.29, then also Σ_W is irreducible. ☐

Lemma 4.39 (Realizations of wheels and whirls)**.** *Let W be a realization of $\mathsf{M} \in \{\mathsf{W}_n, \mathsf{W}^n\}$. Up to scaling $E = S \sqcup R$, W has a basis*

$$w^1 = s_1 + r_1 - t \cdot r_n, \qquad w^i = s_i + r_i - r_{i-1}, \qquad i = 2, \dots, n, \tag{4.11}$$

where $t = 1$ if $\mathsf{M} = \mathsf{W}_n$ and $t \in \mathbb{K}\backslash\{0,1\}$ if $\mathsf{M} = \mathsf{W}^n$. In particular wheels are not binary. For $\mathsf{M} = \mathsf{W}_n$ the cyclic group $\mathbb{Z}_n$ acts on X_W, Σ_W and Δ_W by "turning the wheel".

Proof. Since $S \in \mathcal{B}_{\mathsf{M}}$ we may assume that the coefficients of s_j in w^i form a unit matrix, that is, $w^i_{s_j} = \delta_{i,j}$. The triangle $\{s_j, r_j, s_{j+1}\}$ then forces $w^j_{r_j}, w^{j+1}_{r_j} \neq 0$ and $w^i_{r_j} = 0$ for $i \neq j, j+1$. Suitably scaling $r_1, w^2, r_2, w^3, \dots, r_{n-1}, w^n, s_1, \dots, s_n$ successively yields (4.11). The claim on t follows from $R \in \mathcal{C}_{\mathsf{W}_n}$ and $R \in \mathcal{B}_{\mathsf{W}^n}$ respectively.

For $\mathsf{M} = \mathsf{W}_n$, $\mathbb{Z}_n$ acts on W, hence on ψ_W, hence on X and J_W, hence on Σ_W, and finally on Δ_W by Theorem 4.34. ☐

Proposition 4.40 (Uniqueness of schemes for wheels and whirls)**.** *Let W be a realization of $\mathsf{M} \in \{\mathsf{W}_n, \mathsf{W}^n\}$. In terms of suitable coordinates $z_1, \dots, z_n, y_1, \dots, y_n$ of $\mathbb{K}^E = \mathbb{K}^{S \sqcup R}$, $\psi_W = \det A$ and $M_W = I_{n-1}(A)$ where*

$$A_n := \begin{pmatrix} z_1 & y_1 & 0 & \cdots & \cdots & 0 & y_n \\ y_1 & z_2 & y_2 & 0 & \cdots & \cdots & 0 \\ 0 & y_2 & z_3 & y_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & y_{n-3} & z_{n-2} & y_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & y_{n-2} & z_{n-1} & y_{n-1} \\ y_n & 0 & \cdots & \cdots & 0 & y_{n-1} & z_n \end{pmatrix}.$$

In particular X_W, Σ_W and Δ_W depend only on n up to isomorphism.

Proof. We may assume that W be the realization from Lemma 4.39. Denote the variables corresponding to $r_1, \dots, r_n$ and $s_1, \dots, s_n$ by $z'_1, \dots, z'_n$ and $y_1, \dots, y_n$ respectively. Consider the linear automorphism of $\mathbb{K}^E = \mathbb{K}^{S \sqcup R}$ defined by

$$z_1 := z'_1 + y_1 + t^2 \cdot y_n, \qquad z_i := z'_i + y_i + y_{i-1},$$

for $i = 2, \dots, n$. Then Q_W is represented by the matrix

$$\begin{pmatrix} z_1 & -y_1 & 0 & \cdots & \cdots & 0 & -t \cdot y_n \\ -y_1 & z_2 & -y_2 & 0 & \cdots & \cdots & 0 \\ 0 & -y_2 & z_3 & -y_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -y_{n-3} & z_{n-2} & -y_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & -y_{n-2} & z_{n-1} & -y_{n-1} \\ -t \cdot y_n & 0 & \cdots & \cdots & 0 & -y_{n-1} & z_n \end{pmatrix}.$$

Suitable scaling of $y_1, \dots, y_n$ turns this matrix into A_n. □

Corollary 4.41 (Small wheels and whirls)**.** *Theorem 4.37 holds for* $\mathsf{M} = \mathsf{W}_3$ *and* $\mathsf{M} = \mathsf{W}^n$ *for* $n \leqslant 4$.

Proof. By Proposition 4.40, we may assume that $M_W = I_{k+1}(A_n)$ where $k = n - 2$. Further we are free to extend the field $\mathbb{K}$. Consider the morphism of algebraic varieties of matrices

$$Y := \mathbb{K}^{n \times k} \to \left\{ A \in \mathbb{K}^{n \times n} \mid A = A^t,\ \operatorname{rk} A \leqslant k \right\} =: Z, \quad B \mapsto BB^t.$$

Let $y_{i,j}$ and $z_{i,j}$ be the coordinates on Y and Z respectively. Then $\Delta_W = V(I_{n-1}(A_n))$ identifies with $V(z_{1,3}, z_{2,4}) \subseteq Z$ for $n = 4$ and with Z itself for $n \leqslant 3$. Both the preimage Y of Z and for $n = 4$ the preimage

$$V(y_{1,1}y_{1,3} + y_{1,2}y_{2,3}, y_{2,1}y_{1,4} + y_{2,2}y_{2,4})$$

of $V(z_{1,3}, z_{2,4})$ are irreducible. It thus suffices to show that Y surjects onto Z, which holds for all $k \leqslant n$.

Let $A \in Z$ and $I \subseteq \{1, \dots, n\}$. Assume that $\operatorname{rk} A = |I| = k$ with rows $i \in I$ of A linearly independent. Apply row operations C to make the rows $i \notin I$ of CA zero. Then CAC^t is non-zero only in rows and columns $i \in I$. Modifying C to include further row operations turns CAC^t into a diagonal matrix. Extending $\mathbb{K}$ by square roots if necessary, we can write $CAC^t = D^2$ where D has exactly k non-zero diagonal entries. Then $A = BB^t$ where $B := C^{-1}D$ considered as an element of Y by dropping zero columns. □

Lemma 4.42 (Operations on wheels and whirls)**.** *Let* $\mathsf{M} \in \{\mathsf{W}_n, \mathsf{W}^n\}$.

(a) The bijection (see (2.5))

$$S \sqcup R = E \xrightarrow{\nu} E^\vee = R \sqcup S, \quad s_i \leftrightarrow r_i, \quad r_i \leftrightarrow s_i,$$

identifies $\mathsf{M} = \mathsf{M}^\perp$.

(b) *Unless n is minimal, the handle partition of $\mathsf{M}\backslash s_i$ consists of non-disconnective handles: the 2-handle $\{r_{i-1}, r_i\}$ and singletons.*
(c) *Unless n is minimal, $\mathsf{W}_n\backslash s_n/r_n = \mathsf{W}_{n-1}$ and $\mathsf{W}^n\backslash s_n/r_n = \mathsf{W}^{n-1}$.*

Proof.
(a) The self-duality claim is obvious (see [Oxl11, Prop. 8.4.4]).
(b) This follows from the description of connectedness in terms of circuits (see (2.2) and Example 2.20).
(c) The operation $\mathsf{M} \mapsto \mathsf{M}\backslash s_n/r_n$ deletes the triangle $\{s_{n-1}, r_{n-1}, s_n\}$ and maps the triangle $\{s_n, r_n, s_1\}$ to $\{s_{n-1}, r_{n-1}, s_1\}$ (see (2.2) and (2.4)). By duality, it acts on triads in the same way (see (a) and (2.6)). The claim then follows from the characterization of wheels and whirl by triangles and triads (see [Sey80, (6.1)]). ☐

Lemma 4.43 (Induction on wheels and whirls)**.** *Theorem 4.37 for $\mathsf{M} = \mathsf{W}_n$ and $\mathsf{M} = \mathsf{W}^n$ follows from the cases $n = 3$ and $n \leqslant 4$ respectively.*

Proof. Write M_n for W_n and W^n respectively. Let W' be any realization of M/r_n. Then $W'\backslash s_n$ is a realization of $\mathsf{M}/r_n\backslash s_n = \mathsf{M}\backslash s_n/r_n = \mathsf{M}_{n-1}$ by Lemma 4.42.(c). By induction hypothesis and Corollary 4.26, $\Sigma_{W'\backslash s_n}$ is irreducible with generic point in $\mathbb{T}^{E\backslash\{s_n,r_n\}}$. By Lemma 4.29, $\Sigma_{W'}$ is then irreducible with generic point in $\mathbb{T}^{E\backslash\{r_n\}}$.

By Lemma 4.42.(b) and Corollary 4.24, $\Sigma_{W\backslash s_n}$ at most one generic point $\mathfrak{q}' \in V(y_{n-1}, y_n)$ while all the other generic points lie in $\mathbb{T}^{E\backslash\{s_n\}}$. By Corollary 4.15, the latter identify with generic points of $\Sigma_{(W\backslash s_n)^\perp}$ in $\mathbb{T}^{E\backslash\{r_n\}}$. Then $W' := (W\backslash s_n)^\perp$ is a realization of $(\mathsf{M}\backslash s_n)^\perp = \mathsf{M}^\perp/s_n = \mathsf{M}/r_n$ (see (2.6) and Lemma 4.42.(a)). By the above, $\Sigma_{W'}$ is irreducible with generic point in $\mathbb{T}^{E\backslash\{r_n\}}$. Thus, $\Sigma_{W\backslash s_n}$ has exactly one generic point $\mathfrak{q}$ in $\mathbb{T}^{E\backslash\{s_n\}}$.

By Lemma 4.29 and Corollary 4.26, Σ_W has then at most two generic points, both in $\mathbb{T}^E$. Assume that there are exactly two such generic points $\mathfrak{p}$ and $\mathfrak{p}'$. Again by Lemma 4.29 we may assume that $\sqrt{\bar{\mathfrak{p}}} = \mathfrak{q}$ and $\sqrt{\bar{\mathfrak{p}}'} = \mathfrak{q}'$ where $\bar{I} = (I + \langle x_n\rangle)/\langle x_n\rangle$.

First suppose $\mathsf{M} = \mathsf{W}_n$ with $n \geqslant 4$. By Lemma 4.39, the cyclic group $\mathbb{Z}_n$ acts on $\{\mathfrak{p}, \mathfrak{p}'\}$ by "turning the wheel". If it acts identically, then $\sqrt{\mathfrak{p}' + \langle x_i\rangle} \supseteq \langle y_{i-1}, y_i\rangle$ for all $i = 1, \ldots, n$ and hence

$$\sqrt{\mathfrak{p}' + \langle x_1, \ldots, x_n\rangle} = \langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle.$$

Then $\operatorname{height}(\mathfrak{p}' + \langle x_1, \ldots, x_n\rangle) = 2n$ which implies $\operatorname{height}\mathfrak{p}' \geqslant n > 3$ by Lemma 4.1.(b), contradicting Theorem 4.23. Otherwise the generator of $\mathbb{Z}_n$ switches the assignment $\mathfrak{p} \mapsto \mathfrak{q}$ and $\mathfrak{p} \mapsto \mathfrak{q}'$ and $n = 2m$ must be even. Then $\sqrt{\mathfrak{p} + \langle x_{2i}\rangle} \supseteq \langle y_{2i-1}, y_{2i}\rangle$ for all $i = 1, \ldots, m$ and hence

$$\sqrt{\mathfrak{p} + \langle x_2, x_4, x_6, \ldots, x_n\rangle} \supseteq \langle x_2, x_4, x_6, \ldots, x_n, y_1, \ldots, y_n\rangle.$$

This leads to a contradiction as before.

Now suppose $\mathsf{M} = \mathsf{W}^n$ with $n \geqslant 5$. For $i = 1, \dots, n$ denote by $\mathfrak{q}_i$ and $\mathfrak{q}'_i$ the generic points of $\Sigma_{W\setminus s_i}$. By the pigeonhole principle, one of $\mathfrak{p}$ and $\mathfrak{p}'$, say $\mathfrak{p}$, is assigned to $\mathfrak{q}'_i$ for 3 spokes s_i. In particular $\mathfrak{p}$ is assigned to $\mathfrak{q}'_i$ and $\mathfrak{q}'_j$ for two non-adjacent spokes s_i and s_j. Then

$$\sqrt{\mathfrak{p} + \langle x_i, x_j\rangle} \supseteq \langle x_i, x_j, y_{i-1}, y_i, y_{j-1}, y_j\rangle.$$

This leads to the contradiction as before.

It follows that Σ_W is irreducible as claimed. $\square$

Theorem 4.37 proves the "only if" part of the following conjecture.

Conjecture 4.44 (Irreducible Jacobian scheme and 3-connectedness)**.** Let M be a matroid of rank $\operatorname{rk}\mathsf{M} \geqslant 2$ on E. Then M is 3-connected if and only if, for some/any realization $W \subseteq \mathbb{K}^E$ of M, both schemes Σ_W and $\Sigma_{W^\perp}$ are irreducible.

5. Examples

In this section we illustrate our results with examples of prism, whirl and uniform matroids.

Example 5.1 (Prism matroid). Consider the prism matroid M (see Definition 2.18) with its unique realization W (see Lemma 2.19). Then

$$\psi_W = x_1x_2(x_3+x_4)(x_5+x_6)+x_3x_4(x_1+x_2)(x_5+x_6)+x_5x_6(x_1+x_2)(x_3+x_4)$$

by Example 3.8. By Lemma 4.27, Δ_W has the unique generic point

$$\langle x_1 + x_2, x_3 + x_4, x_5 + x_6\rangle$$

in $\mathbb{T}^6$. By Corollary 4.24, there can be at most 3 more generic points symmetric to

$$\left\langle x_1, x_2, \psi_{W\setminus\{1,2\}}\right\rangle = \langle x_1, x_2, x_3x_4x_5 + x_3x_4x_6 + x_3x_5x_6 + x_4x_5x_6\rangle.$$

Over $\mathbb{K} = \mathbb{F}_2$ their presence is confirmed by a `Singular` (see [Dec+18]) computation. It reveals a total of 7 embedded points in Σ_W. There are 3 symmetric to each of

$$\langle x_3, x_4, x_5, x_6\rangle \quad \text{and} \quad \langle x_1, x_2, x_3 + x_4, x_5 + x_6\rangle$$

plus $\langle x_1, \dots, x_6\rangle$. However Σ_W is not reduced at any generic point. Since the above associated primes are geometrically prime, the conclusions remain valid over any field $\mathbb{K}$ with $\operatorname{ch}\mathbb{K} = 2$.

A `Singular` calculation over $\mathbb{Q}$ shows that Σ_W has exactly the above associated points for any field $\mathbb{K}$ with $\operatorname{ch}\mathbb{K} = 0$ or $\operatorname{ch}\mathbb{K} \gg 0$. We believe that this holds in fact for $\operatorname{ch}\mathbb{K} \neq 2$.

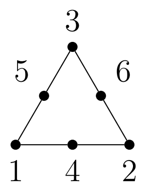
To verify at least the presence of the these associated points in Σ_W for $\operatorname{ch}\mathbb{K} \neq 2$, we claim that

$$\begin{aligned}\langle x_1, x_2, \psi_{W\setminus\{1,2\}}\rangle &= J_W \colon 2((x_3+x_4)x_5^2 - (x_3+x_4)x_6^2),\\ \langle x_3, x_4, x_5, x_6\rangle &= J_W \colon 2(x_1+x_2)^2 x_4 x_6,\\ \langle x_1, x_2, x_3+x_4, x_5+x_6\rangle &= J_W \colon 2x_2(x_3+x_4)x_6^2,\\ \langle x_1, \dots, x_6\rangle &= J_W \colon 2(x_1+x_2)(x_3+x_4)x_6.\end{aligned}$$

The colon ideals on the right hand side can be read off from a suitable Gröbner basis (see [GP08, Lems. 1.8.3, 1.8.10 and 1.8.12]). Using `Singular` we compute such a Gröbner basis over $\mathbb{Z}$ which confirms our claim. There are no odd prime numbers dividing its leading coefficients. It is therefore a Gröbner basis over any field $\mathbb{K}$ with $\operatorname{ch}\mathbb{K} \neq 2$ and the argument remains valid. ◇

Example 5.2 (Whirl matroid). Consider the whirl matroid W^3 (see Example 2.20). It is realized by 6 points in $\mathbb{P}^2$ with the collinearities shown in Figure 3. Since M contracts to the uniform matroid $\mathsf{U}_{2,4}$, M

FIGURE 3. Points in $\mathbb{P}^2$ defining the whirl matroid W^3.



is not regular (see [Oxl11, Thm. 6.6.6]). The configuration polynomial reflects this fact. Using the realization from Lemma 4.39 with $t = -1$, we find

$$\begin{aligned}\psi_W &= x_1x_2x_3 + x_1x_3x_4 + x_2x_3x_4 + x_1x_2x_5 + x_2x_3x_5 + x_1x_4x_5\\ &\quad + x_2x_4x_5 + x_3x_4x_5 + x_1x_2x_6 + x_1x_3x_6 + x_1x_4x_6 + x_2x_4x_6\\ &\quad + x_3x_4x_6 + x_1x_5x_6 + x_2x_5x_6 + x_3x_5x_6 + 4x_4x_5x_6.\end{aligned}$$

Replacing in ψ_W the coefficient 4 of $x_4x_5x_6$ by a 1 yields the matroid polynomial ψ_{M} (see Remark 3.6).

By Theorem 4.23, the configuration hypersurface X_W defined by ψ_W has 3-codimensional non-smooth locus. Using `Singular` (see [Dec+18]) we find a Gröbner basis over $\mathbb{Z}$ of the ideal of partials of ψ_{M}. The only prime numbers dividing leading coefficients are 2, 3 and 5. For $\operatorname{ch}\mathbb{K} \neq 2, 3, 5$ it is therefore a Gröbner basis over $\mathbb{K}$. From its leading exponents one computes that the non-smooth locus of the hypersurface defined by ψ_{M} has codimension 4 (see [GP08, Cor. 5.3.14]). By further `Singular` calculations, this codimension is 4 for $\operatorname{ch}\mathbb{K} = 2, 5$ and 3 for $\operatorname{ch}\mathbb{K} = 3$. ◇

Example 5.3 (Uniform rank-3 matroid). Suppose that $\operatorname{ch}\mathbb{K} \neq 2, 3$. Then the configuration $W = \langle w^1, w^2, w^3\rangle \subseteq \mathbb{K}^3$ defined by

$$(w^i_j)_{i,j} = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 6 & 12 \end{pmatrix}$$

realizes the uniform matroid $\mathsf{U}_{3,6}$ (see Example 2.16). The entries of Q_W with indices (i,j) where $i \leqslant j$ are linearly dependent (see Remark 3.22). By Lemma 3.24, ψ_W thus depends on fewer than 6 variables. More precisely, a `Singular` calculation shows that Σ_W has Betti numbers $(1, 5, 10, 10, 5, 1)$, is not reduced and hence not Cohen–Macaulay.

Now, take W' to be a generic realization of $\mathsf{U}_{3,6}$. Then the entries of $Q_{W'}$ with indices (i,j) where $i \leqslant j$ are linearly independent (see [BCK16, Prop. 6.4]), $\Sigma_{W'}$ is reduced Cohen–Macaulay with Betti numbers $(1, 6, 8, 3)$. So basic geometric properties of the configuration hypersurface X_W are not determined by the matroid M, but depend on the realization W. ◇

Example 5.4 (Uniform rank-2 matroid). Suppose that $\operatorname{ch}\mathbb{K} \neq 2$ and consider the uniform matroid $\mathsf{U}_{2,n}$ for $n \geqslant 3$ (see Example 2.1). A realization W of $\mathsf{U}_{2,n}$ is spanned by two vectors $w^1, w^2 \in \mathbb{K}^n$ for which (see Example 2.16)

$$c_{W,\{i,j\}} = \det \begin{pmatrix} w^1_i & w^1_j \\ w^2_i & w^2_j \end{pmatrix}^2 \neq 0,$$

for $1 \leqslant i < j \leqslant n$. Then

$$\psi_W = \sum_{1 \leqslant i < j \leqslant n} c_{W,\{i,j\}} \cdot x_i \cdot x_j,$$

and the ideal J_W is generated by n linear forms. These forms may be written as the rows of the Hessian matrix

$$H_W := H_{\psi_W} = (c_{W,\{i,j\}})_{i,j},$$

where by convention $c_{W,\{i,i\}} = 0$. Since uniform matroids are connected, Theorem 4.23 implies that H_W has rank exactly 3.

For $n \geqslant 4$, this amounts to a classical-looking linear algebra fact: suppose that $A = (a^2_{i,j})_{i,j} \in \mathbb{K}^{n\times n}$ is a matrix with squared entries. Then its 4×4 minors are zero provided that the numbers $a_{i,j}$ satisfy the Plücker relations defining the Grassmannian $\mathrm{Gr}_{2,n}$. An elementary direct proof was shown to us by Darij Grinberg (see [Gri18]). ◇

References

[AGV18] Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant. *Log-Concave Polynomials I: Entropy and a Deterministic Approximation Algorithm for Counting Bases of Matroids.* 2018. eprint: arXiv:1807.00929.

[AM11a] Paolo Aluffi and Matilde Marcolli. "Algebro-geometric Feynman rules". In: *Int. J. Geom. Methods Mod. Phys.* 8.1 (2011), pp. 203–237.

[AM11b] Paolo Aluffi and Matilde Marcolli. "Feynman motives and deletion-contraction relations". In: *Topology of algebraic varieties and singularities.* Vol. 538. Contemp. Math. Amer. Math. Soc., Providence, RI, 2011, pp. 21–64.

[BB03] Prakash Belkale and Patrick Brosnan. "Matroids, motives, and a conjecture of Kontsevich". In: *Duke Math. J.* 116.1 (2003), pp. 147–188.

[BCK16] Cristiano Bocci, Enrico Carlini, and Joe Kileel. "Hadamard products of linear spaces". In: *J. Algebra* 448 (2016), pp. 595–617.

[BEK06] Spencer Bloch, Hélène Esnault, and Dirk Kreimer. "On motives associated to graph polynomials". In: *Comm. Math. Phys.* 267.1 (2006), pp. 181–225.

[BH93] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings.* Vol. 39. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993, pp. xii+403.

[BK97] D. J. Broadhurst and D. Kreimer. "Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops". In: *Phys. Lett. B* 393.3-4 (1997), pp. 403–412.

[Bro15] Francis Brown. *Periods and Feynman amplitudes.* 2015. eprint: arXiv:1512.09265.

[Bro17] Francis Brown. "Feynman amplitudes, coaction principle, and cosmic Galois group". In: *Commun. Number Theory Phys.* 11.3 (2017), pp. 453–556.

[BW10] Christian Bogner and Stefan Weinzierl. "Feynman graph polynomials". In: *Internat. J. Modern Phys. A* 25.13 (2010), pp. 2585–2618.

[CH96] Collette R. Coullard and Lisa Hellerstein. "Independence and port oracles for matroids, with an application to computational learning theory". In: *Combinatorica* 16.2 (1996), pp. 189–208.

[Dec+18] Wolfram Decker et al. Singular — *A computer algebra system for polynomial computations.* Version 4-1-1. 2018.

[Dor11] Dzmitry Doryn. "On one example and one counterexample in counting rational points on graph hypersurfaces". In: *Lett. Math. Phys.* 97.3 (2011), pp. 303–315.

[Eis95] David Eisenbud. *Commutative algebra*. Vol. 150. Graduate Texts in Mathematics. With a view toward algebraic geometry. New York: Springer-Verlag, 1995, pp. xvi+785.

[GP08] Gert-Martin Greuel and Gerhard Pfister. *A* **Singular** *introduction to commutative algebra*. extended. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann, With 1 CD-ROM (Windows, Macintosh and UNIX). Springer, Berlin, 2008, pp. xx+689.

[Gri18] Darij Grinberg. *A symmetric bilinear form and a Plücker identity*. Nov. 6, 2018. MathOverflow: a/314720. URL: https://mathoverflow.net/a/314720.

[HC52] D. Hilbert and S. Cohn-Vossen. *Geometry and the imagination*. Translated by P. Neményi. Chelsea Publishing Company, New York, N. Y., 1952, pp. ix+357.

[HS06] Craig Huneke and Irena Swanson. *Integral closure of ideals, rings, and modules*. Vol. 336. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 2006, pp. xiv+431.

[Kut74] Ronald E. Kutz. "Cohen-Macaulay rings and ideal theory in rings of invariants of algebraic groups". In: *Trans. Amer. Math. Soc.* 194 (1974), pp. 115–129.

[Mar10] Matilde Marcolli. *Feynman motives*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010, pp. xiv+220.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*. Second. Vol. 8. Cambridge Studies in Advanced Mathematics. Translated from the Japanese by M. Reid. Cambridge University Press, Cambridge, 1989, pp. xiv+320.

[OT92] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*. Vol. 300. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Berlin: Springer-Verlag, 1992, pp. xviii+325.

[Oxl11] James Oxley. *Matroid theory*. Second. Vol. 21. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2011, pp. xiv+684.

[Pat10] Eric Patterson. "On the singular structure of graph hypersurfaces". In: *Commun. Number Theory Phys.* 4.4 (2010), pp. 659–708.

[Piq19] Matthieu Piquerez. *A multidimensional generalization of Symanzik polynomials*. 2019. eprint: arXiv:1901.09797.

[Sey80] P. D. Seymour. "Decomposition of regular matroids". In: *J. Combin. Theory Ser. B* 28.3 (1980), pp. 305–359.

[Sta18] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2018.

[Tru92] K. Truemper. *Matroid decomposition*. Academic Press, Inc., Boston, MA, 1992, pp. x+398.

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