THE COMBINATORIAL LAPLACIAN OF THE TUTTE COMPLEX

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ABSTRACT. Let M be an ordered matroid, and $C_{\bullet\bullet}(M)$ be an exterior algebra over its underlying set E, graded both by corank and nullity. Then $C_{\bullet 0}(M)$ is the simplicial chain complex of $IN(M)$, the simplicial complex whose simplices are indexed by the independent sets of the matroid. Dually, $C_{0\bullet}(M)$ is the cochain complex of $IN(M^*)$. We give a combinatorial description of a basis of eigenvectors for the combinatorial Laplacian of a family of boundary maps on the double complex, extending work by Kook, Reiner, and Stanton [11] on $IN(M)$. The eigenvalues are enumerated by a weighted version of the Tutte polynomial, using an identity of Etienne and Las Vergnas [8]. As an application, we prove a duality theorem for the cohomology of Orlik-Solomon algebras.

1. SUMMARY

Throughout this note, let M be a matroid whose underlying set E is totally ordered. Let $L(M)$ be the lattice of flats of M, and ρ its rank function. In his survey [1], Björner examines a simplicial complex $IN(M)$ comprised of the independent sets of M, and he relates its singular homology to that of the order complex of the dual matroid, M^*.

In [11], Kook, Reiner, and Stanton continue Björner's study by explicitly determining the spectrum of the combinatorial Laplace operator on the chain complex of $IN(M)$. Recall that, if $\partial_p : C_p \to C_{p-1}$ is the boundary map in the chain complex $C_\bullet$, and each C_p is a vector space with a positive definite inner product, then the combinatorial Laplacian is defined to be

$$\Delta_p = \partial_p^t \partial_p + \partial_{p+1} \partial_{p+1}^t.$$

They show that, for the complex they consider and a particular inner product, the eigenvalues of the Laplacian are nonnegative integers obtained from the cardinalities of the flats of M.

The purpose of this note is to extend their work in two different directions. On one hand, let R be an integral domain, and $a : E \to R^\times$ a function that we shall interpret as an assignment of weights from the units of R to the set E. Extend a additively to all subsets of E. This choice of weights gives a choice of inner products, and it turns out that the eigenvalues of the generalized Laplacian are, more generally, the weights of the flats of M. The inner product involved is closely related to the one introduced for hyperplane arrangements by Schechtman and Varchenko in [15], then generalized to arbitrary matroids by Brylawski and Varchenko in [3].

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On the other hand, let $\Lambda(M)$ be the exterior algebra with base ring R over the set E. $\Lambda(M)$ has a bigrading that reflects the structure of the matroid: let

$$\Lambda_p^q(M) = \langle e_1 \wedge e_2 \wedge \cdots \wedge e_q : \rho(\{e_1, \ldots, e_q\}) = p \rangle . \tag{1}$$

The usual differential on the exterior algebra makes $\Lambda(M)$ into a double complex. The subcomplex $\Lambda_p^p(M)$, generated by monomials given by independent sets, is equivalent to the singular chain complex of $IN(M)$ that Kook, Reiner, and Stanton consider. Our second objective is to extend their description of the spectrum of the Laplacian of $IN(M)$ to what we shall call here the Tutte complex, $\Lambda_p^q(M)$ (Theorem 20).

Perhaps the most natural motivation for this extension is that one finds the matroid and its dual play symmetric roles in the larger complex, via a pairing between $\Lambda(M)$ and $\Lambda(M^*)$. Also, since the dimensions of the spaces $\Lambda_p^q(M)$ determine the Tutte polynomial, a generating function for the spectrum of the Laplacian is a refinement of both the Tutte polynomial and Kook, Reiner, and Stanton's spectrum polynomial $\mathrm{Spec}_M(t, q)$.

The last application given here is to a homological property of the Orlik-Solomon algebra. Various authors have considered the cohomology of the Orlik-Solomon algebra $\mathbf{A}(M)$ of a hyperplane arrangement M, viewed as a cochain complex under a boundary map $d(x) = \omega \wedge x$, where

$$\omega = \sum_{H \in E} a(H)H,$$

for some complex- or integer-valued weight function a: see, for example, [7, 16, 17]. Gelfand and Zelevinsky [10] have shown that

$$\cdots \longrightarrow \Lambda_p^{p+1} \longrightarrow \Lambda_p^p \longrightarrow \mathbf{A}^p(M) \longrightarrow 0$$

is a free resolution of the Orlik-Solomon algebra. In this context, the pairing between $\Lambda(M)$ and $\Lambda(M^*)$ gives an isomorphism in cohomology for all p (Theorem 26):

$$H^p(\mathbf{A}^\bullet(M), d) \cong H^{m-2n+p}(\mathbf{A}^\bullet(M^*), d),$$

where m and n are the cardinality and rank of the matroid M, respectively.

2. Laplacians

This section begins by defining the generalized Laplacians studied here (2.1), then indicates the relation between the complex of a matroid and that of the matroid's dual (2.2). The combinatorial fact from 2.3, together with a discussion of the kernel of the Laplacian (2.4), lead to a characterization of the Laplacian's spectrum, Theorem 20.

2.1. **Maps.** Facts about matroids used here can be found in the book by Oxley.[14] Fix an integral domain R, and let M be an ordered matroid. Call the generators of the R-exterior algebra $e_1, \ldots, e_m$, where e_i corresponds to the ith element of E in its given total order, and $m = |E|$. In order to write the monomial basis efficiently, for any subset $S = \{s_1, s_2, \ldots, s_q\}$ of E ordered so that $s_i < s_{i+1}$ for all i, let

$$e_S = s_1 \wedge s_2 \wedge \cdots \wedge bs_q.$$

We first introduce boundary maps that make the exterior algebra into a double complex. The usual boundary map $\partial = \partial(M)$ in the exterior algebra is defined for all $S \subseteq E$ by

$$\partial(e_S) = \sum_{s \in S} \varepsilon_S(s) e_{S-\{s\}},$$

where $\varepsilon_S(s) = (-1)^{k-1}$, if s is the kth element of S. Then $\partial(M)$ is the sum of two boundary maps $\partial_h = \partial_h(M)$ and $\partial_v = \partial_v(M)$. $\partial_h : \Lambda^q_p(M) \to \Lambda^{q-1}_{p-1}(M)$ is given by restricting the sum to those $s \in S$ for which $\rho(S - \{s\}) < \rho(S)$, while $\partial_v : \Lambda^q_p(M) \to \Lambda^{q-1}_p(M)$ sums over the remaining elements of S.

Fix a weight function $a : E \to R^\times$ on the underlying set of the matroid, and define an R-module map $\phi : \Lambda^q(M) \to \Lambda^q(M)$ by

$$\phi(e_1 \wedge e_2 \wedge \cdots \wedge e_q) = a(e_1)^{-1} a(e_2)^{-1} \cdots a(e_q)^{-1} e_1 \wedge e_2 \wedge \cdots \wedge e_q. \tag{2}$$

For subsets $T \subseteq S \subseteq E$, let $\varepsilon_S(T) = (-1)^k$, where k is least number of transpositions required to sort the list of increasing elements of S, followed by increasing elements of $T - S$, into the order given by E. Then let τ denote the map of R-modules $\tau : \Lambda^q(M) \to \Lambda^{m-q}(M)$ given by $\tau(e_S) = e_{E-S}$ on monomials e_S. This gives a bilinear form on $\Lambda^q_p(M)$, for each p, q, defined by $(u, v) = \det(\tau\phi(u) \wedge v)$. The bilinear form is symmetric, since

$$(e_S, e_T) = \begin{cases} \prod_{s \in S} a(s)^{-1} & \text{if } S = T \\ 0 & \text{otherwise.} \end{cases}$$

Next, let δ, δ_h, and δ_v be the adjoints to ∂, ∂_h, and ∂_v, respectively, with respect to $(\cdot, \cdot)$. By direct calculation, $\delta_v : \Lambda^q_p(M) \to \Lambda^{q+1}_p(M)$ satisfies

$$\delta_v(e_S) = \sum_{s \in V} a(s) s \wedge e_S = \sum_{s \in V} \varepsilon_{S \cup \{s\}}(s) a(s) e_{S \cup \{s\}},$$

where $V = \{s \in E : \rho(S \cup \{s\}) = \rho(S)\}$, while δ_h is given by the same expression, with V replaced by $E - V$, and $\delta = \delta_h + \delta_v$, acting by left-multiplication by the element

$$\omega = \sum_{s \in E} a(s) s.$$

Finally, define the operators $\Delta_h = \Delta_h(M)$ and $\Delta_v = \Delta_v(M)$ by setting

$$\Delta_h = \delta_h \circ \partial_h + \partial_h \circ \delta_h, \quad \text{and} \quad \Delta_v = \delta_v \circ \partial_v + \partial_v \circ \delta_v.$$

When the base ring is the real numbers $\mathbf{R}$ and all of the weights are positive, the inner product is positive definite. Then Δ_v and Δ_h are combinatorial Laplace operators in the traditional sense. Most of the ideas presented here are essentially independent of the base ring, however, so we shall work with an arbitrary integral domain R while still calling the operators above "Laplacians."

2.2. **Duality.** In order to simplify notation (in the long run), $\Lambda^q_p(M)$ will be re-graded to reflect the symmetry between M and its dual. Let

$$C_{pq}(M) = \Lambda^{n-p+q}_{n-p}(M), \tag{3}$$

where n is, as usual, the rank of the matroid M. The basic properties of $C_{\bullet\bullet}(M)$ can now be expressed in the following way.

4. Proposition. *Let $C_{\bullet\bullet}(M)$ be the bigraded free R-module defined above, for a matroid of rank n and cardinality m. Let $a : E \to R^\times$.*

1. *Any of the four pairs of maps $\{\partial_h, \delta_h\} \times \{\partial_v, \delta_v\}$ make $C_{\bullet\bullet}(M)$ a double complex.*

2. *For all $0 \leq p \leq n$ and $0 \leq q \leq m-n$, there is a nondegenerate pairing*

$$C_{pq}(M) \otimes C_{qp}(M^*) \to R$$

given by $\langle u, v\rangle = \det(\phi(u) \wedge v)$, where ϕ, depending on the weight function, is defined in (2). *Under this pairing, $\partial_h(M)$ is adjoint to $\delta_v(M^*)$, $\delta_h(M)$ is adjoint to $\partial_v(M^*)$, as well as $\Delta_h(M)$ to $\Delta_v(M^*)$ and $\Delta_v(M)$ to $\Delta_h(M^*)$.*

Proof. To show that $C_{\bullet\bullet}(M)$ is a double complex under each choice of horizontal and vertical boundary maps, one must verify that

$$\partial_h\partial_v + \partial_v\partial_h = 0 \quad \text{and} \quad \delta_h\partial_v + \partial_v\delta_h = 0.$$

The remaining pair of identities follows by taking adjoints with respect to $(\cdot,\cdot)$.

To prove the second claim, let $U \subseteq E$ be a set of rank $n-p$ in M and cardinality $n-p+q$, corresponding to a monomial $e_U \in C_{pq}(M)$. Let ρ^* denote the rank function of M^*. Since $\rho^*(E-U) = m-n-q$ and the cardinality of $E-U$ is $m-n+p-q$, the monomial e_{E-U} lies in $C_{qp}(M^*)$. Since $a(e)$ was taken to be a unit in R for each $e \in E$, ϕ is an isomorphism, and so is the map $C_{pq}(M) \to \mathrm{Hom}_R(C_{qp}(M^*), R)$ defined by $u \mapsto \langle u, -\rangle$.

The identities $\langle\partial_h(u), v\rangle = \langle u, \delta_v(v)\rangle$ and $\langle\delta_h(u), v\rangle = \langle u, \partial_v(v)\rangle$, together with the two remaining choices obtained by exchanging h and v, all follow immediately from the formulas of Section (2.1). The adjunctions of the Laplacians are obtained by combining the identities above. ∎

Equivalently, the bilinear forms $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ induce isomorphisms

$$C_{pq}(M) \longrightarrow \mathrm{Hom}_R(C_{qp}(M^*), R) \longrightarrow C_{qp}(M^*),$$

respectively. The composition $\phi^{-1}\tau$ gives isomorphisms of double complexes,

$$\phi^{-1}\tau : (C_{pq}(M), b_h, b_v) \to (C_{qp}(M^*), b_v^t, b_h^t), \tag{5}$$

where $\{b_h, b_h^t\} = \{\partial_h, \delta_h\}$, and $\{b_v, b_v^t\} = \{\partial_v, \delta_v\}$. These are obtained by composing the

Extend the weight function to each subset U of E by setting

$$a(U) = \sum_{e\in U} a(e).$$

6. Corollary. *For any matroid and weight function a,*

$$\Delta_h + \Delta_v = a(E)\cdot \mathrm{Id}.$$

Proof. Let $\Delta = \delta\partial + \partial\delta$ be the Laplace operator on the whole exterior algebra. It follows from Proposition 4(1) that $\Delta = \Delta_h + \Delta_v$. On the other hand, it is easy to check that $\Delta(x) = a(E)x$ for any $x \in \Lambda(M)$. ∎

The generating function

$$s_M(x,y) = \sum_{p,q} \dim\, C_{pq}(M) x^p y^q \tag{7}$$

is called the corank-nullity polynomial, and it is well known that $s_M(x-1, y-1) = t_M(x,y)$, the Tutte polynomial. For more information about this family of matroid invariants, refer to [5], [1], or the comprehensive treatment in [4].

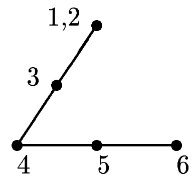

FIGURE 1. Affine representation of a matroid M

2.3. **A Matroid Identity.** In our discussion of eigenvalues, we shall require a formula due to Kook, Reiner, and Stanton [12] that expresses the corank-nullity polynomial in terms of characteristic polynomials of flats and their duals. Recall that the characteristic polynomial of a matroid M is given by $\chi(M;t) = \sum_{X\in L(M)} \mu(\hat{0}, X) t^{\rho(M)-\rho(X)}$.

Let us recall the terminology of activities.[5] Let M be an ordered matroid, and S an independent set of M. Let $X = [S]$, the smallest flat containing S. An element $e \in X - S$ is *externally active* for S (in M) if it is the least element in the (unique) circuit contained in $S \cup \{e\}$. Dually, for any set $S \subseteq E$, not necessarily independent, let $X = [S]$. An element $e \in S$ is *internally active* in S if e is externally active in $X - S$ in the matroid $M(X)^*$.

Etienne and Las Vergnas [8] have proven the following.

8. Theorem (Theorem 4.2, [8])**.** *Let M be an ordered matroid. Every set $S \subseteq E$ can be written uniquely as a disjoint union $S = \pi_1(S) \cup \pi_2(S)$ with the properties that*

1. *$\pi_1(S)$ is an independent set in $M/[\pi_2(S)]$ with no externally active elements, and*
2. *$\pi_2(S)$ has no internally active elements.*

We remark that Kook, Reiner, and Stanton give an explicit algorithm in [11, Theorem 1] to find this decomposition when S is a base of M. It is not difficult to check that their algorithm applies without change in this more general case, although we shall not require it in what follows.

Recall that an independent set S in M is said to have no broken circuits if it has no externally active elements.[1] Write $\mathbf{nbc}(M)$ for the set of all such independent sets S in M, although other authors have used this notation to refer to just the bases of M.

9. Corollary. *Let M be an ordered matroid. There is a bijection*

$$f : 2^E \to \coprod_{X\in L(M)} \mathbf{nbc}(M/X) \times \mathbf{nbc}(M(X)^*),$$

where $M(X)$ denotes the restriction of M to X, and $M(X)^*$ its dual. The map is given by $f(S) = (\pi_1(S), X - \pi_2(S))$, where $X = [\pi_2(S)]$, and the decomposition $S = \pi_1(S) \cup \pi_2(S)$ is the one from Theorem 8.

10. Example. Consider the elements $S = \{1,2,4,5\}$ of the matroid M given by Figure 2.3. Since none of $\{1,2,4\}$ are internally active and 5 is not externally active in $M/\{1,2,3,4\}$, it follows that $\pi_1(S) = \{5\}$ and $\pi_2(S) = \{1,2,4\}$. Since $X = \{1,2,3,4\}$, we have $f(S) = (\{5\}, \{3\})$. Conversely, $\mathbf{nbc}(M/X) = \{\emptyset, \{5\}\}$, and

$$\mathbf{nbc}(M(X)^*) = \{\emptyset, \{1\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}\};$$

eleven other sets $S \subseteq E$ map under f into $\mathbf{nbc}(M/X) \times \mathbf{nbc}(M(X)^*)$.

The theorem gives the generating function identity from [12] that underlies the structure of the Laplacian's eigenspaces (Theorem 20).

11. Corollary ([12]). *Let M be a matroid of rank n. Then*

$$s_M(x,y) = \sum_{X \in L(M)} (-1)^{n-|X|} \chi(M/X; -x)\chi(M(X)^*; -y).$$

Proof. For any ordering of a matroid M, the coefficient of t^p in $(-1)^n\chi(M,-t)$ counts the number of independent sets of rank $\rho(M) - p$, with external activity zero. [1, (7.4.2)] The corank-nullity polynomial $s_M(x,y)$ is a generating function for all subsets of the underlying set. Keeping track of the grading on both sides of the bijection of Corollary 9 establishes the identity. ∎

2.4. **Eigenspaces.** We begin examining the eigenspaces of the Laplacian with the zero eigenspace. In the real, positive-definite case, the kernel of the Laplacian has a natural interpretation as homology, which we defer to section 3.3. Here, we give an explicit basis for a submodule $A_p(M)$ of the kernel of Δ_v. Under a genericity assumption on the weights, we show later that the containment of $A_p(M)$ in the kernel is an isomorphism.

The first step is to recall the definition of the Brylawski and Varchenko's flag complex $\mathbf{F}^\bullet(M)$ from [3]. The basis is the set of flags of flats $(F^0, F^1, \ldots, F^p)$, where $\hat{0} = F^0 < F^1 < \ldots < F^p$, and $\rho(F^i) = i$, modulo the relations

$$\sum_{\tilde{F^i}} (F^0, \ldots, \tilde{F^i}, \ldots, F^p) = 0$$

for each $1 \leq i \leq p$. Note that $\mathbf{F}^p(M) \cong H_p^W(L(M))$, the Whitney homology of the order complex of M.[13] Define a map $B^p : \mathbf{F}^p \to C_{n-p,0}$ for $F = (F^0, \ldots, F^p)$ by

$$B(F) = \sum_{U \in \mathbf{S}} a(s_1)a(s_2)\cdots a(s_p)e_U, \tag{12}$$

where $\mathbf{S}$ consists of sets $U = \{s_1, \ldots, s_p\}$ for which $s_i \in F^i$ and $s_i \notin F^{i-1}$, for $1 \leq i \leq p$. This map lifts Brylawski and Varchenko's map in [3] from the Orlik-Solomon algebra to the exterior algebra; however, their proof that (12) is well-defined remains valid here.

13. Definition. For $0 \leq p \leq n$, let $A_p(M) = \operatorname{im} B^{n-p} \subseteq C_{p0}(M)$. Its properties are summarized by the next proposition:

14. Proposition.

1. *$A_\bullet(M)$ is a chain complex under the restriction of both boundary maps ∂_h and δ_h.*
2. *$A_\bullet(M)$ lies in the zero eigenspace of $\Delta_v(M)$ and the $a(E)$-eigenspace of $\Delta_h(M)$.*
3. *Dually, $A_\bullet(M^*) \hookrightarrow C_{0\bullet}(M)$ is in the $a(E)$-eigenspace of $\Delta_v(M)$.*

Proof. The first claim is proven by noting that $\mathbf{F}^\bullet(M)$ is a (co)chain complex with two boundary maps, and $B : \mathbf{F}^\bullet(M) \to C_{n-\bullet,0}(M)$ is a chain map: see [15].

The two parts of the second claim are equivalent, by Corollary 6. To prove the first part, it is sufficient to show that $\delta_v B(F) = 0$ for any flag $F \in \mathbf{F}^p(M)$. Explicitly,

$$\delta_v B(F) = \sum_{\substack{U \in \mathbf{S} \\ s \in E-U}} a(s)a(s_1)a(s_2)\cdots a(s_p)e_s \wedge e_U,$$

where $\mathbf{S}$ is the defined for equation (12) and $s \in X^i$ but $s \notin X^{i-1}$ for some $1 \leq i \leq p$. By exchanging s with s_i, we see that each term in the sum occurs twice, with opposite signs.

The injection in the third claim comes from restricting the isomorphism $\tau\phi$, from (5). That $A_\bullet(M^*)$ is in the $a(E)$-eigenspace of $\Delta_v(M)$ follows by combining this with part (2). ∎

Recall that the characteristic polynomial is a generating function for the dimensions of $\mathbf{F}^{n-\bullet}(M)$.[1] We show that $B^p : \mathbf{F}^p \to A_{n-p}$ is an isomorphism, in order to conclude that

$$\sum_{p=0}^{n} \dim(A_p(M))t^p = (-1)^n \chi(M; -t). \tag{15}$$

In fact, the next proposition makes a slightly stronger statement.

16. Proposition. *For any matroid M and weight function $a : E \to R^\times$, the map $B^{n-p} : \mathbf{F}^{n-p}(M) \to A_p$ is an isomorphism, and the inclusion $A_p \hookrightarrow C_{p0}(M)$ has a splitting.*

Proof. Only a sketch is given, since the proof is substantially the same as that of Theorem 14 in [11]. $\mathbf{F}^p(M)$ has a basis consisting of flags of the form

$$F = (\hat{0}, [s_p], [s_p, s_{p-1}], \ldots, [s_p, \ldots, s_1]),$$

where $s_1 < s_2 < \cdots s_p$ are the elements of an independent set with external activity zero.[1] Let V denote the submodule of $C_{n-p,0}(M)$ generated by monomials e_S for which S has external activity zero. Then $\dim V = \dim \mathbf{F}^p(M)$. Let $\pi : C_{n-p,0}(M) \to V$ be the projection map that kills all other monomials. It is not hard to check that, on a flag F from the basis above, $\pi \circ B : \mathbf{F}^p(M) \to V$ satisfies

$$\pi B(F) = a(s_1)\cdots a(s_p)e_S,$$

where s_i is the least element of $F^i - F^{i-1}$, for $1 \leq i \leq p$. Since the weights $a(s)$ are units in R, πB is a R-module isomorphism. It follows that $B^p : \mathbf{F}^p(M) \to C_{n-p,0}(M)$ is an injection, and has a splitting. ∎

2.5. **The main result.** We have shown that the kernel of the Laplacian is isomorphic to the flag complex. All of the eigenspaces of Δ_v can be described in a similar way, and the description is the main result of this note, Theorem 20. We begin by showing, as in [11], that the Laplacian obeys the Leibniz rule on certain subspaces.

17. Lemma. *Let X be a flat of a matroid M. Let $e_S \in C_{\bullet\bullet}(M/X)$ and $e_T \in C_{\bullet\bullet}(X)$ be monomials for which S is independent in M/X, and T spans X. Then*

$$\Delta_v(M)(e_S \wedge e_T) = \Delta_v(M/X)(e_S) \wedge e_T + e_S \wedge \Delta_v(X)(e_T).$$

Proof. First check that

$$\delta_v(M)(e_S \wedge e_T) = \delta_v(M/X)(e_S) \wedge e_T + (-1)^{|S|} e_S \wedge \delta_v(X)(e_T). \tag{18}$$

From the description of the maps in Section 2.1,

$$\begin{aligned}\delta_v(M)(e_S \wedge e_T) &= \sum_{e \in V \cap (E-X)} a(e) e \wedge e_S \wedge e_T + \sum_{e \in V \cap X} a(e) e_S \wedge e_T \\ &= \sum_{e \in V \cap (E-X)} a(e) e \wedge e_S \wedge e_T + (-1)^{|S|} e_S \wedge \sum_{e \in V \cap X} a(e) e \wedge e_T,\end{aligned}$$

where

$$V = \{e : \rho_M(S \cup T \cup \{e\}) = \rho_M(S \cup T)\}.$$

Since T spans X, $\rho_{M/X}(U) = \rho_M(U \cup T) - \rho_M(X)$ for any subset $U \subseteq E - X$, so

$$V \cap (E - X) = \{e : \rho_{M/X}(S \cup \{e\}) = \rho_{M/X}(S)\}.$$

Again, since T spans X, $V \cap X = X$. Claim (18) follows.

Using a similar argument, one finds that

$$\partial_v(M)(e_S \wedge e_T) = \partial_v(M/X)(e_S) \wedge e_T + (-1)^{|S|} e_S \wedge \partial_v(X)(e_T). \tag{19}$$

The lemma is proven by combining (18) and (19). ∎

Recall that $A_\bullet(M)$ (from Definition 13) is the kernel of $\Delta_v(M)$.

20. Theorem. *Let M be a matroid and $L(M)$ its lattice of flats. Let $a : E \to R^\times$ be a weight function.*

1. *For a flat X of M, the image of the injection*
$$\sigma_X : A_p(M/X) \otimes A_q(M(X)^*) \to C_{pq}(M)$$
defined on monomials by $\sigma_X(e_S \otimes e_T) = e_S \wedge e_{X-T}$ is the $a(X)$-eigenspace of $\Delta_v(M)$ on $C_{pq}(M)$. If R is a field, all eigenvectors of Δ_v appear in this way.
2. *The images of σ_X and σ_Y are orthogonal with respect to the inner product $(\cdot,\cdot)$, for flats $X \neq Y$.*
3. *The map*
$$\sigma : \bigoplus_{X \in L(M)} A_p(M/X) \otimes A_q(M(X)^*) \to C_{pq}(M),$$
where $\sigma = \bigoplus_{X \in L(M)} \sigma_X$, is an injection. Over the fraction field of R, σ is an isomorphism.

A flat X is said to be cyclic if $M(X)$ contains no ithsmuses. Note that we may restrict the sum above to cyclic flats, for if $M(X)$ contains an ithsmus, $A_\bullet(M(X)^*) = 0$.

Proof. Clearly each map σ_X is an injection. We show first that the image of σ_X is indeed contained in the $a(X)$-eigenspace of Δ_v. Note that the R-modules $A_p(M/X) \otimes A_q(M(X)^*)$ themselves do not have monomial bases; strictly speaking, then, σ_X is defined by restriction from $C_{p0}(M/X) \otimes C_{q0}(M(X)^*)$. Let $u = \sum_{S \in \mathbf{S}} c_S e_S$ and $v = \sum_{T \in \mathbf{T}} c'_t e_T$ be arbitrary elements of $A_p(M/X)$ and $A_q(M(X)^*)$, respectively. Since $A_p(M/X)$ is contained in the independence complex $C_{p0}(M/X)$, each $S \in \mathbf{S}$ is independent in M/X. For the same reason, each $T \in \mathbf{T}$ is independent in the matroid $M(X)^*$, which means that $X - T$ spans $M(X)$. Therefore $\sigma_X(u \otimes v)$ satisfies the conditions of Lemma 17. By Proposition 14, u is a zero eigenvector of

$\Delta_v(M/X)$, and v is a $a(X)$-eigenvector of $\Delta_v(M(X)^*)$. It follows from Lemma 17 that $\sigma_X(u \otimes v)$ is a $0 + a(X)$ eigenvector of $\Delta_v(M)$, as required.

In order to prove Assertion 2, let

$$P = \mathbf{Z}[\{b_s, b_s^{-1} : s \in E\}], \tag{21}$$

the ring of Laurent polynomials in the weights. Under the natural weight function $a : E \to P$, the eigenvalues are all distinct. Since Δ_v is self-adjoint with respect to $(\cdot,\cdot)$, the $a(X)$- and $a(Y)$-eigenspaces are orthogonal. Any weight function factors through P, and the orthogonality is preserved by a change of rings.

It follows that we have an injection

$$\sigma : \bigoplus_{X \in L(M)} A_p(M/X) \otimes A_q(M(X)^*) \hookrightarrow C_{pq}(M).$$

Counting ranks using Corollary 11 and (15) proves Assertion 3. That is, if R is a field, then the map is an isomorphism. ∎

Over a field, the map σ is an isomorphism. In general, though, the eigenspaces of Δ_v do not span the operator's domain. The gap between the two is addressed by the remark following Theorem 26.

In order to recover the results of [11], which we claim to have generalized, let $q = 0$ and $a : E \to \mathbf{R}$ by $a(s) = 1$ for all $s \in E$. For any matroid M of rank n and cardinality m,

$$A_0(M) \cong \mathbf{F}^n(M) = H_n^W(M) \cong H_{n-2}(L(M)) \cong H_{m-n-1}(IN(M^*)),$$

from [1]. Since $(C_{n-p,0}(M), \partial_h)$ is isomorphic to the singular chain complex of $IN(M)$, and its Laplacian Δ_h satisfies $\Delta_h = m \cdot \mathrm{Id} - \Delta_v$, Theorem 20 gives the eigenspace decomposition of Δ_h.

3. Interpretation

3.1. **Generating Functions.** One can form a generating function that encodes only the eigenvalues of Δ_v and their multiplicities, using Theorem 20. Let $\mathbf{b} = \{b_s : s \in E\}$ be a set of indeterminates, and write $b_X = \prod_{s \in X} b_s$. Set

$$\Phi_M(x, y, \mathbf{b}) = \sum_{\substack{p,q \in Z \\ X \in L(M)}} c_{pq}(X) x^p y^q b_X$$

where the coefficient $c_{pq}(X)$ is the dimension of the $a(X)$-eigenspace of $\Delta_{v;pq}$. By Theorem 20, all of the eigenvalues have this form. Φ_M has some immediate properties:

1. $\Phi_M(x, y, \mathbf{b}) = \sum_{X \in L(M)} (-1)^{n-|X|} \chi(M/X; -x) \chi(M(X)^*; -y) b_X$;
2. $\Phi_M(x, y, \mathbf{b}) = b_E \Phi_{M^*}(y, x, \{b_e^{-1}\})$;
3. $\Phi_M(x, 0, \{b_e \leftarrow q\}) = x^n \mathrm{Spec}_M(x^{-1}, q)$.

The first restates Theorem 20, while the second uses Proposition 4(2) and Corollary 6. The last statement follows from Corollary 17 of [11].

22. Example. Let M be the matroid of Figure 2.3. The table below lists the cyclic flats X, together with the characteristic polynomials of $M(X)^*$ and M/X.

X	$(-1)^{\rho(M(X)^*)}\chi(M(X)^*,-y)$	$(-1)^{\rho(M/X)}\chi(M/X,-x)$
$\emptyset$	1	$4+8x+5x^2+x^3$
$\{1,2\}$	$1+y$	$2+3x+x^2$
$\{4,5,6\}$	$1+y$	$1+x$
$\{1,2,3,4\}$	$2+3y+y^2$	$1+x$
$\{1,2,3,4,5,6\}$	$4+8y+5y^2+y^3$	1

The first identity indicates how to calculate $\Phi_M(x,y,\mathbf{b})$:

$$\begin{aligned}\Phi_M(x,y,\mathbf{b}) = \left(x^3+5x^2+8x+4\right) + \left(x^2y+x^2+3xy+3x+2y+2\right) b_1b_2 + \\ + (xy+x+y+1)\, b_4b_5b_6 + \left(xy^2+3xy+y^2+2x+3y+2\right) b_1b_2b_3b_4 + \\ + \left(y^3+5y^2+8y+4\right) b_1\cdots b_6.\end{aligned}$$

By specializing according to identity 3, we find that the spectrum polynomial

$$t^n\mathrm{Spec}_M(t^{-1},q) = \sum_{\lambda,p} \dim(\Delta_{v;p0}(M))_\lambda t^i q^\lambda$$

equals

$$4+8t+5t^2+t^3+(2+3t+t^2)q^2+(1+t)q^3+(2+2t)q^4+4q^6,$$

where Δ_λ denotes the λ-eigenspace of an operator Δ.

The next theorem refines well-known formulas for the corank-nullity and Tutte polynomials.

23. Theorem. *For any ordered matroid M,*

1. $\Phi_M(x,y,\mathbf{b}) = \sum_{S\subseteq E} x^{\mathrm{cor}(S)}y^{\mathrm{nul}(S)}b_X$, *where* $\mathrm{cor}(S)=n-\rho(S)$, $\mathrm{nul}(S)=|S|-\rho(S)$, *and* $X=[\pi_2(S)]$ *(from Theorem 8).*
2. $\Phi_M(x-1,y-1,\mathbf{b}) = \sum_{\text{bases } B \text{ of } M} x^{i(B)}y^{e(B)}b_{[\pi_2(B)]}$, *where* $i(B)$ *and* $e(B)$ *are, respectively, the number of internally and externally active elements of* B.

Proof. The first follows by comparing the first identity in this section with Corollary 9. The second is an application of [8, Corollary 5.4]. ∎

3.2. **Reconstruction.** For a flat X of M, let $\alpha^*(X)=\dim H_{\rho(X)-1}(IN(X))$. In [11], Kook, Reiner, and Stanton ask to what extent $\mathrm{Spec}_M(t,q)$ determines $t_M(x,y)$, since

$$t^n\mathrm{Spec}_M(-t^{-1},q) = \sum_{X\in L(M)} (-1)^{n-\rho(X)}\alpha^*(X)q^{|X|}\chi(M/X;t),$$

while the polynomial

$$\overline{\chi}_M(t,q) = \sum_{X\in L(M)} q^{|X|}\chi(M/X;t)$$

equals $t_M(x,y)$ under a change of variables. Brylawski calls $\overline{\chi}_M(t,q)$ the Poincaré polynomial of M; see [4].

While it is not known whether or not $\mathrm{Spec}_M(t,q)$ determines the Tutte polynomial, it is relatively easy to see that the weighted version of the question has an affirmative answer.

24. Proposition. *Let M be a matroid with underlying set E. From $\Phi_M(x,0,\mathbf{b})$ one can determine the list of flats of M, hence the isomorphism class of the matroid.*

Proof. We have

$$\Phi_M(x,0,\mathbf{b}) = \sum_{X\in L(M)} (-1)^{n-\rho(X)}\alpha^*(X)\chi(M/X;-x)b_X.$$

Recall that $\chi(M/X;-x)=0$ only if M/X contains loops and $\alpha^*(X)=0$ only if X contains an isthmus. Since M/X never contains a loop, the product b_X gives the members of each cyclic flat X. The degree of $\chi(M/X;-x)$ provides the rank of X. Using [14, Ex. 13, p. 78], one can reconstruct the matroid from this information. ∎

3.3. **Cohomology of the Orlik-Solomon Algebra.** This section applies the results of Section 2 to interpret the homology of the Tutte complex.

The first step is the definition of the Orlik-Solomon algebra of a matroid M: let $\mathbf{A}^{n-p}(M) = C_{p0}/\partial_v(C_{p1})$, the quotient of independent p-tuples by the boundaries of circuits. $\mathbf{A}^\bullet(M)$ is a (co)chain complex in two ways, with boundary maps induced by both ∂_h and δ_h. Its algebra structure is inherited from the exterior algebra. The Orlik-Solomon algebra was introduced originally in [2] and [13] to describe the cohomology ring of the complex complement of a hyperplane arrangement, though many of its interesting properties extend to arbitrary matroids. [9, 3, 1]

By this definition, $\mathbf{A}^{n-p}(M)$ is the zero homology module of the chain complex $(C_{p\bullet}(M),\partial_v)$. More generally, we have the following proposition, which was proven for hyperplane arrangements by Gel'fand and Zelevinsky. [10]

25. Proposition.

$$\cdots \xrightarrow{\partial_v} C_{p1}(M) \xrightarrow{\partial_v} C_{p0}(M) \longrightarrow \mathbf{A}^{n-p}(M) \longrightarrow 0$$

is a free resolution of the Orlik-Solomon algebra of M, as a chain or cochain complex. Dually,

$$\cdots \xrightarrow{\delta_h} C_{1q}(M) \xrightarrow{\delta_h} C_{0q}(M) \longrightarrow \mathbf{A}^{m-n-q}(M^*) \longrightarrow 0$$

is also a free resolution.

Proof. We show that the first sequence above is exact at $C_{\bullet q}(M)$ for $q>0$. It is sufficient to do so for the natural weight function $a: E\to P$, where P is the ring of Laurent polynomials over E. For short, let $H_q = H_q(C_{p\bullet},\partial_v)$. Since δ_v is a chain homotopy for the chain complex (C_{pq},∂_v), Δ_v induces the zero map on the homology module H_q. Therefore $\det(\Delta_v)$ annihilates H_q. Theorem 20(1) shows that, when $q>0$,

$$\det\Delta_v = \prod_{X\in L(M)-\{\emptyset\}} a(X)^{k_X}$$

for some nonnegative integers k_X. Since ∂_v does not depend on the weight function, though, $\det(\Delta_v)H_q=0$ implies that $H_q=0$, as required.

To show that the second sequence is exact for each q, $0 \le q \le m-n$, use (5): $\phi^{-1}\tau$ is an isomorphism, hence a quasiisomorphism:

$$(C_{\bullet q}(M), \partial_v) \overset{\phi^{-1}\tau}{\cong} (C_{q\bullet}(M^*), \delta_h).$$

∎

If the weight function a is real and positive, then Δ_v is a Laplacian in the strict sense, and has the well-known property that its kernel is isomorphic to the homology of the complex. Note that in general, however, the kernel of Δ_v is larger, if the weight function satisfies $a(X) = 0$ for a flat $X \neq \emptyset$.

The homology of the total complexes $(C_{\bullet\bullet}(M), \partial_h, \partial_v)$ and $(C_{\bullet\bullet}, \delta_h, \delta_v)$ are zero, since they are the reduced cellular (co)chain complexes of a solid simplex with vertices E. Let us consider the other two pairs of boundary maps.

Let $d : \mathbf{A}^p \to \mathbf{A}^{p+1}$ denote the boundary map induced by δ_h, and let $a : E \to R^\times$ be a weight function of a matroid M. Suppose that $R = \mathbf{C}$ and that M is realizable over $\mathbf{C}$. Then $H^p(\mathbf{A}^\bullet(M), d)$ has an interpretation as cohomology of sections of a line bundle, parameterized by the weight function, over the complement of the hyperplanes in complex space; refer to [17] or [7, 16]. In that case, the following theorem could be proven geometrically.

26. Theorem. *For any matroid M, of rank n and cardinality m, there is an isomorphism*

$$H^{n-p}(\mathbf{A}^\bullet(M), d) \cong H^{m-n-p}(\mathbf{A}^\bullet(M^*), d). \tag{27}$$

Proof. By Proposition 25,

$$\begin{aligned} H_p^{tot}(C_{\bullet\bullet}(M), \partial_v, \delta_h) &= H^{n-p}(\mathbf{A}^\bullet(M), d) \\ &\cong H^{m-n-p}(\mathbf{A}^\bullet(M^*), d), \end{aligned}$$

taking homology with respect to ∂_v followed by δ_h first, then taking homology in the reverse order and using (5). ∎

It should be mentioned that the theorem cannot be extended to more general weight functions: $a : E \to R$ with images not in $R^\times$. For example, let $R = \mathbf{Z}[\{b_s : s \in E\}]$ and consider the natural weight function $a : E \to R$. Then Eisenbud, Popescu, and Yuzvinsky [6, Theorem 3.1] have shown, in the context of hyperplane arrangements, that $H^p(\mathbf{A}^\bullet(M), d) \cong 0$ for all $p \neq n$, and that the module $H^n(\mathbf{A}^\bullet(M), d)$ has projective dimension n over R. Furthermore, they show that the complex $(\mathbf{A}^{n-\bullet}(M), d)$ is in fact a minimal free resolution over R of the top cohomology module. In this case, then, the isomorphism (27) will fail. However, by localizing R to invert the weights, one obtains the natural weight function function $a : E \to P$ (from (21)). Then Theorem 26 applies to their result to show that

$$H^n(\mathbf{A}^\bullet(M), d) \cong H^{m-n}(\mathbf{A}^\bullet(M^*), d)$$

is a P-module whose support has codimension β, Crapo's beta-invariant, and

$$H^{n-p}(\mathbf{A}^\bullet(M), d) \cong H^{m-n-p}(\mathbf{A}^\bullet(M^*), d) \cong 0$$

for $p \neq 0$.

We make a final remark. Let

$$C^0_{pq}(M) = \bigoplus_{X \in L(M)} A_p(M/X) \otimes A_q(M(X)^*).$$

For all X at least one of the complexes $(A_\bullet(M/X), \delta_h)$ and $(A_\bullet(M(X)^*), \delta_h)$ is exact, so the total complex of $(C^0_{pq}(M), \partial_v, \delta_h)$ is also exact. The long exact sequence of

$$0 \longrightarrow C^0_{\bullet\bullet}(M) \xrightarrow{\sigma} C_{\bullet\bullet}(M) \longrightarrow \text{coker}\,(\sigma) \longrightarrow 0$$

shows that, for all p,

$$H^{tot}_p(\text{coker}\,(\sigma), \overline{\partial_v}, \overline{\delta_h}) \cong H^{tot}_p(C_{\bullet\bullet}(M), \partial_v, \delta_h)$$
$$\cong H^{n-p}(\mathbf{A}^\bullet(M), d).$$

In particular, for weight functions $a : E \to R^\times$ that give the Orlik-Solomon algebra nonzero cohomology groups, the inclusion σ of the Laplacian's eigenspaces into the Tutte complex is not an isomorphism.

In this case, it should be possible to relate the cokernel of σ to Brylawski, Schechtman, and Varchenko's determinant formulas in [3, 15].

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References

[1] Anders Björner. The homology and shellability of matroids and geometric lattices. In *Matroid applications*, volume 40 of *Encyclopedia Math. Appl.*, pages 226–283. Cambridge Univ. Press, Cambridge, 1992.

[2] Egbert Brieskorn. Sur les groupes de tresses [d'après V. I. Arnol'd]. pages 21–44. Lecture Notes in Math., Vol. 317, 1973.

[3] T. Brylawski and A. Varchenko. The determinant formula for a matroid bilinear form. *Adv. Math.*, 129(1):1–24, 1997.

[4] Thomas Brylawski. The Tutte polynomial. I. General theory. In *Matroid theory and its applications*, pages 125–275. Liguori, Naples, 1982.

[5] Thomas Brylawski and James Oxley. The Tutte polynomial and its applications. In *Matroid applications*, volume 40 of *Encyclopedia Math. Appl.*, pages 123–225. Cambridge Univ. Press, Cambridge, 1992.

[6] D. Eisenbud, S. Popescu, and S. Yuzvinsky. Hyperplane Arrangement Cohomology and Monomials in the Exterior Algebra. Preprint, 2000.

[7] H. Esnault, V. Schechtman, and E. Viehweg. Cohomology of local systems on the complement of hyperplanes. *Invent. Math.*, 109(3):557–561, 1992.

[8] Gwihen Etienne and Michel Las Vergnas. External and internal elements of a matroid basis. *Discrete Math.*, 179(1-3):111–119, 1998.

[9] I. M. Gel'fand and G. L. Rybnikov. Algebraic and topological invariants of oriented matroids, 1990.

[10] I. M. Gel'fand and A. V. Zelevinsky. Algebraic and combinatorial aspects of the general theory of hypergeometric functions. *Funct. Anal. Appl.*, 20:183–97, 1986.

[11] W. Kook, V. Reiner, and D. Stanton. Combinatorial Laplacians of matroid complexes. *J. Amer. Math. Soc.*, 13(1):129–148, 2000.

[12] W. Kook, V. Reiner, and D. Stanton. A convolution formula for the Tutte polynomial. *J. Combin. Theory Ser. B*, 76(2):297–300, 1999.

[13] Peter Orlik and Louis Solomon. Combinatorics and topology of complements of hyperplanes. *Invent. Math.*, 56(2):167–189, 1980.

[14] James G. Oxley. *Matroid theory*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1992.

[15] V. V. Schechtman and A. N. Varchenko. Arrangements of hyperplanes and Lie algebra homology. *Invent. Math.*, 106:139–194, 1991.

[16] H. Terao, V. V. Schechtman, and A. N. Varchenko. Local systems over complements of hyperplanes and the Kac-Kazhdan conditions for singular vectors. *J. Pure Appl. Algebra*, 100(1–3):93–102, 1995.

[17] S. Yuzvinsky. Cohomology of the Brieskorn-Orlik-Solomon algebras. *Comm. Algebra*, 23(14):5339–5354, 1995.

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