# Homological aspects of hyperplane arrangements

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**Abstract.** The purpose of this expository article is to survey some results and applications of free resolutions related to hyperplane arrangements. We include some computational examples and open problems.

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## 1. Introduction

One facet of recent work on hyperplane arrangements is the influence of rational homotopy theory and the appearance of some interesting homological algebra. Arrangement complements are formal in the sense of Sullivan, and their cohomology rings are well-understood. On the other hand, the combinatorics of the cohomology ring is quite intricate (see [14]), which leads to some interesting and unsolved problems. Providing a complete survey of the rational homotopy theory of hyperplane arrangements is beyond the scope of these lecture notes; the objective here is instead to use two related topics to give some idea of the existing literature and future directions. Koszul duality plays a somewhat unifying role. Some of the topics here are also discussed in the surveys [21, 47, 19, 14].

These notes are organized as follows. The rest of this section defines graded free resolutions and introduces some Lie algebras associated with a discrete group, in this case the fundamental group of an arrangement complement.

Section §2 considers free resolutions over the arrangement's cohomology ring. If $M$ is the complement of $n$ hyperplanes in $\mathbb{C}^\ell$, let $A = H^\bullet(M,\mathbb{Q})$. We recall a "classical" interpretation of the linear strand of the resolution of the trivial $A$-module in terms of the lower central series of the fundamental group. Both can be understood in combinatorial term in two interesting cases: when $A$ is a Koszul algebra (§2.1) and, more recently, for decomposable arrangements (§2.2). Beyond the linear strand, one encounters the homotopy Lie algebra. Roos [39] has recently provided examples of arrangements for which this Lie algebra is not finitely generated, and for which its Hilbert series is transcendental (§2.3).

Section §3 focusses on resolutions over an exterior algebra. This is a more recent inquiry that begins with work in [41, 18, 42]. Via a standard linear inclusion of an arrangement complement in a torus $T = (\mathbb{C}^*)^n$, the exterior algebra $E = H^\bullet(T, \mathbb{Q})$ acts on $A$ and its vector space dual, $A^* = H_\bullet(M, \mathbb{Q})$. The homology of the arrangement complement, $A^*$, has a remarkable property: its free resolution as an $E$-module is linear (§3.3). We see this provides a link with resonance and cohomology of local systems on arrangement complements, which ties in this second topic with the lectures notes of Falk [19] and Dimca-Yuzvinsky[14].

### 1.1. Free resolutions

Let $\mathbf{k}$ be a field, and let $R$ be a Noetherian $\mathbf{k}$-algebra, graded by the natural numbers. Let $|x|$ denote the degree of a homogeneous element $x \in R$. The algebra $R$ is *graded-commutative* if $xy = (-1)^{|x||y|}yx$ for all homogeneous elements $x, y \in R$. If $M$ is a finitely-generated, graded left $R$-module, for $j \in \mathbb{Z}$, let $M(j)$ denote the module with degrees shifted down by $j$: that is, $M(j)_i = M_{i+j}$, for all $i \in \mathbb{Z}$. A graded free resolution of $M$ is a resolution $(F_\bullet, \partial)$ by graded, free modules in which the differential $\partial$ has degree 0. One can write such a resolution as follows:

$$0 \longleftarrow M \longleftarrow \bigoplus_j R(-j)^{b_{0j}} \xleftarrow{\partial_1} \bigoplus_j R(-j)^{b_{1j}} \cdots \xleftarrow{\partial_i} \bigoplus_j R(-j)^{b_{ij}} \xleftarrow{\partial_{i+1}} \cdots$$

for some integers $\{b_{ij} \colon i, j \in \mathbb{Z}\}$. The resolution is *minimal* if the entries of each $\partial_i$ are contained in the maximal homogeneous ideal $R_+ = \bigoplus_{j>0} R_j$. In this case, the number $b_{0j}$ is just the number of generators of $M$ in degree $j$ in a minimal generating set. Minimal resolutions exist, but they are not unique. However, the numbers $\{b_{ij} \colon i \geq 0, j \in \mathbb{Z}\}$ are the same for every minimal resolution of a module $M$, and are called the bigraded Betti numbers of $M$. Let

$$P(M, s, t) = \sum_{i \geq 0, j} b_{ij} s^i t^j, \tag{1.1}$$

the Poincaré-Betti polynomial of $M$. Computing the Euler characteristic of the resolution in each degree gives an expression for the Hilbert series of $M$:

$$H(M, t)/H(R, t) = P(M, -1, t). \tag{1.2}$$

Suppose $(F_\bullet, \partial)$ is a minimal resolution of $M$. Since $\mathbf{k} = R/R_+$, we get

$$\mathbf{k} \otimes_R F_\bullet = \bigoplus_j \Big( \mathbf{k}(-j)^{b_{0j}} \xleftarrow{0} \bigoplus_j \mathbf{k}(-j)^{b_{1j}} \cdots \xleftarrow{0} \bigoplus_j \mathbf{k}(-j)^{b_{ij}} \xleftarrow{0} \cdots \Big),$$

where the differential is zero. In homology, then, $\dim_{\mathbf{k}} \operatorname{Tor}_i^R(\mathbf{k}, M)_j = b_{ij}$, for all $i, j$. Similarly, $\dim_{\mathbf{k}} \operatorname{Ext}_R^i(M, \mathbf{k})_{-j} = \dim_{\mathbf{k}} \operatorname{Hom}_R(F_i, \mathbf{k})_{-j} = b_{ij}$ for all $i, j$.

Suppose that $R(-j)$ is a summand of $F_i$, for some $j$, $i > 0$. Then since $R$ is nonnegatively graded, its image $\partial_i(R(-j))$ in $F_{i-1}$ is contained in summands

$R(-j')$, where $j' < j$. It follows that, for any strictly increasing sequence of integers $c_0 < c_1 < c_2 < \cdots$, the differential in $F_\bullet$ restricts to the subcomplex

$$\bigoplus_{j \le c_0} R(-j)^{b_{0j}} \longleftarrow \bigoplus_{j \le c_1} R(-j)^{b_{1j}} \longleftarrow \cdots \longleftarrow \bigoplus_{j \le c_i} R(-j)^{b_{ij}} \longleftarrow \cdots .$$

**Definition 1.1.** If $F_\bullet$ is a minimal resolution of $M$, let $d_0$ be the smallest degree of a generator of $M$. Since $b_{0j} = 0$ for $j < d_0$, the smallest nonzero subcomplex of the form above occurs using the sequence $c_i = d_0 + i$ for $i \ge 0$. This subcomplex is called the *linear strand of* $F_\bullet$. If a minimal resolution $F_\bullet$ is equal to its linear strand, it is called a *linear resolution*.

Notice that, if a left module $M$ has a linear resolution, then its Hilbert series determines the Betti numbers in its resolution: from (1.2),

$$\begin{aligned} H(M,t)/H(R,t) &= P(M,-1,t) \\ &= (-t)^{d_0} \sum_{i \ge 0} b_{i,d_0+i}(-t)^i. \end{aligned} \tag{1.3}$$

### 1.2. Fundamental groups and the lower central series

From this point onward, we will take our scalars to be the field $\mathbb{Q}$. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be a central arrangement in $\mathbb{C}^\ell$, and let $M = M(\mathcal{A}) = \mathbb{C}^\ell - \bigcup_{H \in \mathcal{A}} H$ denote the complement of the hyperplanes in affine space. Let $G(\mathcal{A}) = \pi_1(M)$ denote its fundamental group. Such groups have been of interest since the '60s: in particular, if $\mathcal{A}$ is the set of reflecting hyperplanes of a reflection group, then $G(\mathcal{A})$ is a (generalized) pure braid group [8, 22]. A modern survey may be found in [44].

For any group $G$, let $G^{(1)} = G$ and $G^{(i)} = [G, G^{(i-1)}]$ for $i \ge 1$, the lower central series of $G$. The quotient $G^{(i)}/G^{(i+1)}$ is abelian, and their sum

$$\mathrm{gr}_{\mathbb{Q}} G = \bigoplus_{i \ge 1} (G^{(i)}/G^{(i+1)}) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{1.4}$$

forms a Lie algebra over $\mathbb{Q}$ with bracket imposed by the commutator in $G$, called *the rational lower central series Lie algebra* of $G$.

One should regard the Lie algebra $\mathrm{gr}_{\mathbb{Q}} G$ as being a simplified approximation to $G$. Let $G'' = [G', G']$, the second derived subgroup of $G$. The maximal metabelian quotient, $G/G''$, is another approximation of the group $G$. Its lower central series ranks were first considered by Chen [9]. Its lower central series Lie algebra, $\mathrm{gr}_{\mathbb{Q}}(G/G'')$, is called the (rational) *Chen Lie algebra* of $G$: for a modern treatment, see [32].

Returning to the case where $G$ is the fundamental group of a hyperplane complement $M(\mathcal{A})$, we will call these the LCS and Chen Lie algebras of $\mathcal{A}$, respectively.

### 1.3. A third Lie algebra

Now let $X$ be a finite-type CW complex. Dualizing the cohomology product $\cup\colon H^1(X,\mathbb{Q}) \otimes H^1(X,\mathbb{Q}) \to H^2(X,\mathbb{Q})$ gives a map

$$\cup^*\colon H_2(X,\mathbb{Q}) \to H_1(X,\mathbb{Q}) \otimes H_1(X,\mathbb{Q}). \tag{1.5}$$

Let $V = H_1(X,\mathbb{Q})$. Since $\cup$ is skew-commutative, for any $z \in H_2(X,\mathbb{Q})$,

$$\cup^*(z) = \sum_i x_i \otimes y_i - y_i \otimes x_i,$$

for some elements $\{x_i\}, \{y_i\}$ of $V$. One can use the image of $\cup^*$ to make a (non-commutative) algebra generated by the vector space $V$: let

$$U = \mathbb{T}(V)/(\mathrm{im}(\cup^*)), \tag{1.6}$$

the quotient of the tensor algebra on $V$ by the two-sided ideal generated by the image of $\cup^*$. Let $[x,y] = x \otimes y - y \otimes x$ for $x, y \in V$. Then since its relations are generated by brackets, $U$ is the universal enveloping algebra of the Lie algebra generated by $V$. This Lie algebra, denoted by $\mathfrak{h}(X)$, is called the *holonomy Lie algebra of* $X$, introduced in [10]. We will write $U = U(\mathfrak{h}(X))$. If the space $X$ is understood, we will write $\mathfrak{h}$ in place of $\mathfrak{h}(X)$.

More precisely, we define an algebra homomorphism $\nabla\colon U \to U \otimes U$ by letting $\nabla(x) = x \otimes 1 + 1 \otimes x$ for $x \in V$, and extending it multiplicatively. An element $x \in U$ is *primitive* if $\nabla(x) = x \otimes 1 + 1 \otimes x$. Let $\mathcal{P}U$ denote the set of primitive elements of $U$. It is not hard to check that $\mathcal{P}U$ is closed under the bracket operation, so $\mathcal{P}U$ is a Lie algebra. We define $\mathfrak{h}(X) = \mathcal{P}U$. (This is an instance of a more general fact: $U$ has the structure of a cocommutative Hopf algebra over $\mathbb{Q}$ with coproduct $\nabla$. In such a situation, $U$ is always the enveloping algebra of its Lie algebra of primitive elements: see [30, 4])

In the case of hyperplane arrangement complements, $\mathfrak{h}(\mathcal{A}) = \mathfrak{h}(M(\mathcal{A}))$ is called the holonomy Lie algebra of $\mathcal{A}$. Recall that the cohomology algebra of the complement $M(\mathcal{A})$ has a combinatorial presentation as the Orlik-Solomon algebra, $A = E/I$, where $E$ is an exterior algebra on $n$ generators, and $I$ an ideal of relations indexed by circuits.

Let $V^* = A^1$, a $\mathbb{Q}$-vector space with basis $\{e_H \colon H \in \mathcal{A}\}$. Let $\{f_H \colon H \in \mathcal{A}\}$ be the dual basis in $V$. Kohno [26] showed that

$$\mathfrak{h}(\mathcal{A}) = \Big\langle f_H \colon H \in \mathcal{A} \Big| \Big[f_H, \sum_{H'\colon H' < X} f_{H'}\Big] \colon H \in \mathcal{A}, X \in L_2(\mathcal{A}), H < X \Big\rangle. \tag{1.7}$$

In particular, the holonomy Lie algebra depends only on $A_i$ for $i = 0, 1, 2$, which is to say that it is determined completely by the number of hyperplanes and their codimension-2 intersections.

*Example* 1. Let $\mathcal{A}$ be an arrangement of $n$ lines through the origin in $\mathbb{C}^2$. Then $A = E/I$, where $I = ((e_ie_j - e_ie_k + e_je_k)\colon 1 \le i < j < k \le n)$. We can identify $A_2$ with $V^* \otimes V^*/W$, where $W$ is the subspace generated by the elements $e_i \otimes e_j + e_j \otimes e_i$ and $e_i \otimes e_j - e_i \otimes e_k + e_j \otimes e_k$ for all $i, j, k$. Then the cup product $A_1 \otimes A_1 \to A_2$ is the quotient map, and the image of its dual is $W^\perp = \{g \in V \otimes V : g(x) = 0 \text{ for all } x \in W\}$. In this case, it can be checked directly that the elements $\left[f_i, \sum_{j=1}^n f_j\right]$ span $W^\perp$, for $1 \le i \le n$, which recovers the presentation of $\mathfrak{h}(\mathcal{A})$ for this arrangement given by (1.7).

### 1.4. Relating the Lie algebras

The example above motivates a key observation due to Shelton and Yuzvinsky [43], for which we need another definition.

**Definition 1.2.** Suppose $B$ is a nonnegatively graded $\mathbb{Q}$-algebra, finitely generated in degree 1, and not necessarily graded-commutative. Let $V = B_1$, a $\mathbb{Q}$-vector space. Then $B \cong \mathbb{T}(V)/R$, for some ideal of relations $R$. Let $W = R_2$, a subspace of $V \otimes V$. The quadratic dual of $B$ is, by definition, the graded algebra

$$B^! = \mathbb{T}(V^*)/(W^\perp). \tag{1.8}$$

Clearly, quadratic duality is an involution: $(B^!)^! \cong B$. By reviewing the construction (1.6) carefully, one obtains the following.

**Proposition 1.3** ([43]). *For any arrangement $\mathcal{A}$, we have $U(\mathfrak{h}(\mathcal{A})) \cong A^!$.*

On the other hand, the fundamental group $G = G(\mathcal{A})$ of an arrangement is 1-formal (in the sense of Sullivan [45]), from which it follows that $\mathrm{gr}_{\mathbb{Q}}(G) \cong \mathfrak{h}(\mathcal{A})$ as Lie algebras [26]. As an application, we could try to understand the LCS ranks, $\phi_i = \mathrm{rank}(G^{(i)}/G^{(i+1)})$, for $i \ge 1$. According to the Poincaré-Birkhoff-Witt Theorem, the associated graded algebra of an enveloping algebra $U(\mathfrak{g})$ (under the bracket-length filtration) is a polynomial algebra. In particular, the Hilbert series of $U(\mathfrak{g})$ is the same as that of the polynomial algebra $\mathbb{Q}[\mathfrak{g}]$.

Since $\mathfrak{h} \cong \mathrm{gr}_{\mathbb{Q}}(G)$ has $\phi_i$ generators of degree $i$, we obtain the formula

$$H(U(\mathfrak{h}), t) = \prod_{i \ge 1} (1 - t^i)^{-\phi_i}. \tag{1.9}$$

Understanding the LCS ranks $\{\phi_i\}$, then, is equivalent to knowing the Hilbert series of the quadratic dual $A^!$ of the Orlik-Solomon algebra. However, finding an explicit description is an open problem except for special classes of arrangements, as we see below.

*Example* 1 (continued). Continuing the example of $n$ lines in $\mathbb{C}^2$, one can see that the element $z := \sum_{j=1}^n f_i$ is central in $\mathfrak{h}(\mathcal{A})$, and its quotient is just the free Lie algebra on $n-1$ generators, which we will call $\mathfrak{f}_{n-1}$. The quotient is (noncanonically) split, so $\mathfrak{h} \cong \mathfrak{f}_{n-1} \times \mathfrak{f}_1$.

On the level of enveloping algebras, $U(\mathfrak{h})$ is isomorphic to the tensor product of a tensor algebra with $n-1$ generators with a polynomial algebra $\mathbb{Q}[z]$. From (1.9), then

$$\prod_{i \ge 1} (1 - t^i)^{-\phi_i} = \frac{1}{(1-(n-1)t)(1-t)} \tag{1.10}$$

by multiplying the Hilbert series of the tensor algebra with that of the one-variable polynomial algebra.

## 2. Resolutions over the Orlik-Solomon algebra

We first outline some known results about free resolutions over the Orlik-Solomon algebra. We use [7] in particular as a reference for multiplicative structures in homological algebra. Consider the module $A_0 = \mathbb{Q}$, on which the positive degree part of $A$ acts trivially. Recall that the Yoneda product makes the Ext groups $\mathrm{Ext}^\bullet_A(\mathbb{Q},\mathbb{Q})$ into a $\mathbb{Q}$-algebra, graded by cohomological degree. The grading from $A$ induces a second grading on the algebra with $\mathrm{Ext}^p_A(\mathbb{Q},\mathbb{Q})_q = 0$ unless $q \leq -p$. For convenience, we let

$$\mathrm{Ext}^p_A(\mathbb{Q},\mathbb{Q})_{(r)} = \mathrm{Ext}^p_A(\mathbb{Q},\mathbb{Q})_{-p-r} \tag{2.1}$$

for $r \geq 0$, so that $\mathrm{Ext}^\bullet_A(\mathbb{Q},\mathbb{Q})$ is nonnegatively graded, and $\mathrm{Ext}^\bullet_A(\mathbb{Q},\mathbb{Q})_{(0)}$ is a subalgebra of $\mathrm{Ext}^\bullet_A(\mathbb{Q},\mathbb{Q})$. Note that $\dim_{\mathbb{Q}} \mathrm{Ext}^i_A(\mathbb{Q},\mathbb{Q})_{(0)} = b_{ii}$, the $i$th Betti number of the linear strand of a minimal free resolution of $\mathbb{Q}$ over $A$.

Löfwall [28] showed that $\mathrm{Ext}^\bullet_A(\mathbb{Q},\mathbb{Q})_{(0)} \cong A^!$ as algebras. Taking Hilbert series via (1.9), then, we get

$$\sum_{i\geq 0} b_{ii}t^i = \prod_{i\geq 1}(1-t^i)^{-\phi_i}, \tag{2.2}$$

which appeared in [36].

### 2.1. Koszul algebras

This leads to a definition.

**Definition 2.1.** A nonnegatively graded $\mathbb{Q}$-algebra $B$ is *Koszul* if $\mathrm{Ext}^\bullet_B(\mathbb{Q},\mathbb{Q})_{(0)} = \mathrm{Ext}^\bullet_B(\mathbb{Q},\mathbb{Q})$.

There are numerous equivalent formulations: for detail and a more general treatment see [5, 23, 37]. From the discussion above, though, $B$ being Koszul is equivalent to having the inclusion be an isomorphism $B^! \cong \mathrm{Ext}_B(\mathbb{Q},\mathbb{Q})$, as well as to the trivial module $\mathbb{Q}$ having a linear resolution. An important consequence (see (1.3)) is that

$$H(B,t)\cdot H(B^!,-t) = H(\mathbb{Q},t) = 1. \tag{2.3}$$

It follows that a Koszul algebra must be *quadratic*: that is, expressible as a quotient of a polynomial algebra by an ideal of relations generated in degree 2. This is a sufficient condition in the case that the ideal is generated by monomials [1], but in general it is difficult to decide if a given algebra is Koszul. A useful test is the following. An algebra $B$ is Koszul if its ideal of relations possesses a quadratic Gröbner basis (also from [1]; see [23] for further discussion.)

The Koszul property has an interpretation in terms of rational homotopy theory, due to Papadima and Yuzvinsky [35]:

**Theorem 2.2** ([35]). *If $X$ is a connected, finite-type formal space, then $H^\bullet(X,\mathbb{Q})$ is Koszul if and only if the rational completion $X_{\mathbb{Q}}$ is an Eilenberg-Maclane space.*

In the case of Orlik-Solomon algebras, Björner and Ziegler [6] showed that $A(\mathcal{A})$ possesses a quadratic Gröbner basis if and only if $\mathcal{A}$ is a supersolvable arrangement. This leads to the following beautiful result:

**Theorem 2.3** ([20]). *If $\mathcal{A}$ is a supersolvable arrangement of rank $\ell$, then the lower central series ranks are given by the formula*

$$\prod_{i\geq 1}(1-t^i)^{-\phi_i} = \pi(\mathcal{A},-t)^{-1},$$

*where $\pi(\mathcal{A},t)$ is the Poincaré polynomial of the arrangement.*

Furthermore, the Poincaré polynomial of a supersolvable arrangement is known to factor as $\pi(\mathcal{A},t) = (1+m_1t)(1+m_2t)\cdots(1+m_\ell t)$ for certain combinatorially significant positive integers $\{m_i\}$, which makes the right-hand side of the identity above more attractive: see [31].

*Proof.* If $\mathcal{A}$ is supersolvable, the Orlik-Solomon algebra is Koszul, by the remark above. The LCS ranks are given by (2.2); using the Koszul property via (2.3), this generating function equals $H(A,-t)^{-1}$. The argument is completed by recalling that the Poincaré polynomial is the Hilbert series of the Orlik-Solomon algebra. □

*Remark* 2.4. Kohno [27] first established the LCS formula above for reflection arrangements of type $A_\ell$. Falk and Randell [20] extended the formula to a class they called fiber-type arrangements, which Terao [46] found was the same as the supersolvable arrangements.

*Problem* 2.5. Examples of Koszul algebras defined by ideals that do not possess a quadratic Gröbner basis are known: see, for example, [40]. However, no such examples of Orlik-Solomon algebras are known. That is, if $\mathcal{A}$ is an arrangement for which $A(\mathcal{A})$ is Koszul, must $\mathcal{A}$ be supersolvable?

### 2.2. Decomposable arrangements

Papadima and Suciu [34] identified another class of arrangements for which the linear strand of a minimal free resolution has a nice structure, generalizing results of Schenck and Suciu [41]. The decomposable arrangements, defined below, are generally not Koszul, but they too have a LCS formula similar to that of Theorem 2.3.

For each subspace $X \in L_2(\mathcal{A})$, write $\mathfrak{h}_X = \mathfrak{h}(\mathcal{A}_X)$. This is the holonomy Lie algebra of a rank-2 arrangement, which we saw in Example 1. The natural projections $\pi_X\colon \mathfrak{h}(\mathcal{A}) \to \mathfrak{h}_X$ assemble to give a map

$$\pi\colon \mathfrak{h} \to \bigoplus_{X\in L_2(\mathcal{A})} \mathfrak{h}_X.$$

Holonomy Lie algebras are graded by bracket length, so $\mathfrak{h}' = \mathfrak{h}_{\geq 2}$. The restriction

$$\pi'\colon \mathfrak{h}' \to \bigoplus_{X\in L_2(\mathcal{A})} \mathfrak{h}'_X \tag{2.4}$$

is always surjective. The arrangement $\mathcal{A}$ is *decomposable* if the map (2.4) is an isomorphism. It is not hard to check that the degree-2 part, $\pi_2$, is always an isomorphism of vector spaces, but in general $\pi'$ has a nonzero kernel in degrees 3 and higher.

A salient feature of this family of arrangements is an effective test for membership. Papadima and Suciu show that, remarkably, $\pi'$ is an isomorphism if and only if $\pi_3$ is an isomorphism. [34, Theorem 2.4] Accordingly, one can decide if an arrangement is decomposable by counting dimensions in (2.4). Based on our calculations in Example 1, if $X$ is a rank-2 flat, then $\mathfrak{h}'_X = \mathfrak{f}'_{m-1}$, where $m = |\mathcal{A}_X|$. By expanding (1.10), one can compute that $\dim_{\mathbb{Q}}(\mathfrak{h}_X)_3 = m(2-3m+m^2)/3$. Similarly, by expanding the generating function (1.9),

$$\dim_{\mathbb{Q}} \mathfrak{h}_3 = \frac{1}{3}(3b_{33} - 3b_{11}b_{22} + b_{11}^3 - b_{11}).$$

So $\mathcal{A}$ is decomposable if and only if

$$3b_{33} - 3b_{11}b_{22} + b_{11}^3 - b_{11} = \sum_{X \in L_2(\mathcal{A})} m_X(2 - 3m_X + m_X^2), \tag{2.5}$$

where $m_X = |\mathcal{A}_X|$.

Since for decomposable arrangements $\mathfrak{h}'$ is known explicitly, so is a generating function for the lower central series ranks of the fundamental group:

**Theorem 2.6** ([34]). *If $\mathcal{A}$ is a decomposable arrangement, the lower central series ranks of $\pi_1(M(\mathcal{A}))$ are given by the formula*

$$\prod_{i \geq 1}(1-t^i)^{-\phi_i} = (1-t)^{m-n} \prod_{X \in L_2(\mathcal{A})} (1-(m_X-1)t)^{-1},$$

*where $n$ is the number of hyperplanes, $m_X = |\mathcal{A}_X|$, and $m = \sum_{X \in L_2(\mathcal{A})}(m_X - 1)$.*

*Example* 2. The arrangement $X_3$ defined by $Q = xyz(x+y)(x+z)(y+z)$ has three triple points, so the right-hand side of (2.5) equals 18. By a computer calculation with Macaulay 2 [25], the first few Betti numbers of $\mathrm{Ext}_A(\mathbb{Q}, \mathbb{Q})$ are

```
0: 1 6 24 80 240
1: . .  1 12  84
2: . .  .  .   1
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Here we use Macaulay 2 notation: that is, $b_{11} = 6$, $b_{22} = 24$, $b_{33} = 80$, and so on.

The left side of (2.5) equals 18 as well, which means $X_3$ is decomposable. By Theorem 2.6 and (2.2), the Betti numbers of the linear strand above are given by the generating function $(1-2t)^{-3}$.

### 2.3. The homotopy Lie algebra

Above, we saw that an isomorphism $\mathrm{Ext}_A(\mathbb{Q}, \mathbb{Q})_{(0)} \cong U(\mathfrak{h})$ relates the linear strand of a free resolution to the holonomy Lie algebra, and this was particularly satisfactory when $A$ is Koszul. More generally, If $B$ is any graded-commutative $\mathbb{Q}$-algebra, then $\operatorname{Ext}_B(\mathbb{Q},\mathbb{Q})$ is a cocommutative Hopf alegebra, which means the primitive elements $\mathcal{P}\operatorname{Ext}_B(\mathbb{Q},\mathbb{Q})$ form a graded Lie algebra, called the *homotopy Lie algebra* of $B$.

For an arrangement $\mathcal{A}$, let

$$\mathfrak{g} = \mathfrak{g}(\mathcal{A}) = \mathcal{P}\operatorname{Ext}_A(\mathbb{Q},\mathbb{Q}),$$

which we will call the homotopy Lie algebra of the arrangement $\mathcal{A}$. Then $\mathfrak{h} = \mathfrak{g}_{(0)}$: the holonomy Lie algebra is the degree-0 subalgebra of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is bigraded, and it should be mentioned that $\mathfrak{g}$ is not a Lie algebra in the classical sense, but rather a "graded Lie algebra" or Lie superalgebra. This is to say that, for homogeneous elements $x \in \mathfrak{g}_{(p)}$ and $y \in \mathfrak{g}_{(q)}$, the bracket satisfies

$$[x,y] = -(-1)^{pq}[y,x],$$

and the Jacobi identity is replaced by

$$(-1)^{pr}[x,[y,z]] + (-1)^{pq}[y,[z,x]] + (-1)^{qr}[z,[x,y]] = 0$$

where, in addition, $z \in \mathfrak{g}_{(r)}$. Here, we are using the fact that the cohomology ring $A$ is generated in degree 1: the signs in the general case are explained, for example, in [4].

If we let $\phi_{ij} = \dim_{\mathbb{Q}} \mathfrak{g}_{i,(j)}$ for $i,j \geq 0$, then $\phi_{i0} = \phi_i$, and the graded version of (1.9) reads

$$P_A(\mathbb{Q}, st^{-1}, t) = H(U(\mathfrak{g}), s, t) = \prod_{i,j\geq 0} \frac{(1+s^i t^{2j+1})^{\phi_{i,(2j+1)}}}{(1-s^i t^{2j})^{\phi_{i,(2j)}}}, \tag{2.6}$$

using (2.1).

Compared to the holonomy Lie algebra, little is known about the homotopy Lie algebra of an arrangement. On the positive side, the structure of $\mathfrak{g}(\mathcal{A})$ is described in [12] for arrangements obtained by intersecting supersolvable arrangements with certain linear subspaces. The resulting arrangements – a subclass of the hypersolvable arrangements – have a cohomology ring which is a Golod quotient of a Koszul algebra. We refer to [12] for details.

On the other hand, Roos [39] has shown that, for certain arrangements, $\mathfrak{g}(\mathcal{A})$ is badly behaved in two ways. First, $\mathfrak{g}(\mathcal{A})$ need not be finitely generated. In particular, $\mathfrak{g}(X_3)$ is not finitely generated. (Recall that we saw in Example 2 that the $X_3$ arrangement is decomposable, and so its holonomy Lie algebra is certainly finitely-generated.) Roos also has shown that the bigraded Hilbert series (2.6) need not be a rational function:

*Example* 3 ([39]). The arrangement of 8 hyperplanes in $\mathbb{C}^3$ defined by $Q = xyz(x-y)(x-z)(y+z)(2x-y-z)(x-2y-z)$ has a transcendental Hilbert series $P_A(\mathbb{Q}, s, t)$. The Betti numbers of the linear strand are given by

$$H(U(\mathfrak{h}), t) = \left(\frac{1-t}{1-2t}\right)^7 \prod_{i\geq 3} (1-t^i)^{-1}.$$

## 3. Part II: Resolutions over the exterior algebra

Some interesting, related free resolutions can be obtained by comparing a hyperplane complement with a torus, as follows. If $\mathcal{A}$ is an arrangement of $n$ hyperplanes, its complement $M$ can be regarded as the intersection of the torus $T = (\mathbb{C}^*)^n$ with a linear subspace of $\mathbb{C}^n$. Identify $H^*(T, \mathbb{Q})$ with the exterior algebra $E = \Lambda(V)$. Then the inclusion $i\colon M \hookrightarrow T$ induces a surjection in cohomology, $i^*\colon E \to A$, which is intrinsic to the combinatorics of the Orlik-Solomon algebra: see [14] in this volume.

This leads to some homological algebra over $E$. In this section, we will consider resolutions of $A = H^\bullet(M, \mathbb{Q})$ and its dual $A^* = H_\bullet(M, \mathbb{Q})$. By means of a beautiful theorem of Eisenbud, Popescu and Yuzvinsky [18], it turns out that these resolutions are closely related to the phenomenon of resonance discussed in Falk's survey [19].

### 3.1. The resolution of $A$ over the exterior algebra

We begin with a fairly simple but illustrative example.

*Example* 4. For the $X_3$ arrangement, the first few Betti numbers of $\mathrm{Tor}^E(\mathbb{Q}, A)$, or equivalently of $\mathrm{Ext}_E(A, \mathbb{Q})$, are

```
0: 1 .  .  .  .   .   .
1: . 3  6  9 12  15  18
2: . 1  9 33 85 180 336
```

Certainly $A$ does not possess a linear free resolution over $E$. However, there appear to be only two nonzero (interesting) rows: in other words, $b_{ij} = 0$ for $j > i + 2$. This reflects the fact that the Castelnuovo-Mumford regularity of $A$ over $E$ is at most $\ell - 1$; that is, for any arrangement of rank $\ell$, we have $\dim_{\mathbb{Q}} \mathrm{Tor}_i^E(A, \mathbb{Q})_j = 0$ for $j \geq i + \ell$. (See [3, Lemma 2.5].) From the diagram, it also seems to be the case that $b_{i,i+1} = 3i$. We will see why in the continuation of this example. These numbers have a topological interpretation, given in Theorem 3.1, below.

Recall that the Lie algebra $\mathfrak{h}' = \mathfrak{h}_{\geq 2}$ inherits the bracket-length grading of $\mathfrak{h}$; therefore the quotient $\mathfrak{h}'/\mathfrak{h}''$ does as well. A result of Fröberg and Löfwall [24, Theorem 4.1(ii)] applies to show

$$\mathrm{Ext}_E(A, \mathbb{Q})_{(1)} \cong \mathfrak{h}'/\mathfrak{h}''$$

as a graded $S$-module, where $S = U(\mathfrak{h}/\mathfrak{h}') = \mathrm{Ext}_E(\mathbb{Q}, \mathbb{Q})$. Then $\dim_{\mathbb{Q}}(\mathfrak{h}'/\mathfrak{h}'')_i = \dim_{\mathbb{Q}}(\mathfrak{h}/\mathfrak{h}'')_{i+1}$ for $i \geq 1$, where $\mathfrak{h}/\mathfrak{h}''$ is the maximal metabelian quotient of the holonomy Lie algebra. That is, the first row of Betti numbers in the resolution of $A$ can be regarded as the LCS ranks of a Lie algebra.

Once again, the 1-formality of arrangement groups makes it possible to interpret this topologically. Papadima and Suciu show in [32] that, if $G$ is 1-formal, then the lower central series Lie algebra $\mathrm{gr}_{\mathbb{Q}}(G/G'')$ is isomorphic to $\mathfrak{h}/\mathfrak{h}''$. For any arrangement $\mathcal{A}$, then, let $G = \pi_1(M(\mathcal{A}))$, and for $i \geq 1$ let $\theta_i = \dim_{\mathbb{Q}}(\mathrm{gr}_i(G/G''))$, the *Chen ranks*. In [42], Schenck and Suciu prove the following:

**Theorem 3.1** ([42]). *For all $i \geq 2$, $\theta_i = b_{i-1,i}$, where $(b_{ij})$ are the Betti numbers in the resolution of the Orlik-Solomon alegebra $A(\mathcal{A})$ over $E$.*

The proof in [42] is direct, using BGG duality to identify $\mathrm{Ext}_E(A, \mathbb{Q})_{(1)}$ as being a linearization of the Alexander invariant. This brings resonance into play, and we will return to this point in §3.4.

A combinatorial description even of the sublinear strand $b_{i-1,i}$ is unknown in general, but nice formulas exist for two special cases. The first is for decomposable arrangements, which were characterized by an isomorphism (2.4). In this case,

$$\mathfrak{h}'/\mathfrak{h}'' \cong \bigoplus_{X \in L_2(\mathcal{A})} \mathfrak{h}'_X/\mathfrak{h}''_X,$$

from [34, Theorem 6.2], so

$$\theta_k = \sum_{X \in L_2(\mathcal{A})} (k-1)\binom{|X|+k-3}{k} \tag{3.1}$$

for $k \geq 2$, by reducing to Chen's calculation [9] for free groups.

*Example* 4 (continued). We saw that the $X_3$ arrangement is decomposable. The only nonzero terms in (3.1) are contributed by the three triple points, so for $k \geq 1$, the sum (3.1) simplifies to

$$b_{k,k+1} = \theta_{k+1} = 3k,$$

as expected.

The second case is that of graphic arrangements. If $\Gamma$ is a graph with $m$ vertices, then $\mathcal{A}(\Gamma)$ is defined to be the arrangement in $\mathbb{C}^m$ with one hyperplane for each edge: $\{z_i - z_j = 0\colon \{i,j\} \in E(\Gamma)\}$. Then by [41, Lemma 6.9],

$$b_{k,k+1} = k(\kappa_2 + \kappa_3) \tag{3.2}$$

for all $k \geq 2$, where $\kappa_s$ denotes the number of complete subgraphs in $\Gamma$ with $s+1$ vertices. We refer to [41] for further explicit computations of the Betti numbers in this resolution and that of $\mathbb{Q}$ over $A$, which depend on some intricate combinatorics in the change of rings spectral sequence.

*Problem* 3.2. Give a direct, combinatorial interpretation of the integers $\dim_{\mathbb{Q}} \mathrm{Tor}_i^E(A, \mathbb{Q})_j$ using the intersection lattice $L(\mathcal{A})$, for arbitrary $i$ and $j$.

### 3.2. Homology of an arrangement complement

Let $E^*$ denote the $\mathbb{Q}$-dual of $E$. By the universal coefficients theorem, $E^* \cong H_\bullet(T, \mathbb{Q})$. In homology, then, we have an inclusion (of coalgebras)

$$i_*\colon H_\bullet(M, \mathbb{Q}) \hookrightarrow H_\bullet(T, \mathbb{Q}) \cong E^*. \tag{3.3}$$

However, since the torus is an $H$-space, $E^*$ is also an algebra: the exterior algebra on $V^*$. Let $\{e_i : H_i \in \mathcal{A}\}$ denote the standard basis of $V$, identifying it with $H^1(M, \mathbb{Q})$. Let $\{e_i^* : H_i \in \mathcal{A}\}$ be the dual basis.

Poincaré duality in the torus amounts to the following in the exterior algebra. Fix an ordered basis of $V$, or, equivalently, a choice of isomorphism $\det\colon E_n \cong \mathbb{Q}$. Then, for each $p$, there is a vector space isomorphism $\phi\colon E_p \to (E^*)_{n-p}$ given by

$$\phi(x)(y) = \det(xy). \tag{3.4}$$

Then, for all $p$, $0 \le -p \le n$,

$$\begin{aligned} A^*_p &= \mathrm{Hom}_{\mathbb{Q}}(A, \mathbb{Q})_p \\ &\cong \mathrm{Hom}_E(A, E)_{n+p} && \text{by (3.4)}, \\ &\cong (\operatorname{ann} I)_{n+p}, \end{aligned} \tag{3.5}$$

where $\operatorname{ann} I$ denotes the annihilator ideal to the defining ideal $I$ of the Orlik-Solomon algebra.

### 3.3. The resolution of $A^*$ over the exterior algebra

Recall from [14, Section 2] that the ideal $I$ is generated by elements $\partial(e_S)$, where the monomial $e_S$ is indexed by a circuit: that is, a minimal dependent set of hyperplanes. With respect to the graded-lexicographic order, the leading monomial in $\partial(e_S)$ is $e_{S'}$, where, if $i$ is the least index in $S$, then $S' = S - \{i\}$.

Additively, the initial ideal $\mathrm{in}(I) = \mathbb{Q}\{e_T : T \text{ contains a broken circuit}\}$.

The following construction captures the combinatorial nature of squarefree monomial ideals. As before let $E$ be an exterior algebra with generators $\{e_1, e_2, \ldots, e_n\}$. Let $R$ be the polynomial algebra $\mathbb{Q}[x_1, x_2, \ldots, x_n]$.

**Definition 3.3.** Let $\Delta$ be an abstract simplicial complex with vertices $[n] = \{1, 2, \ldots, n\}$. The ideals $I_\Delta$ and $J_\Delta$ of $R$ and $E$, respectively,

$$J_\Delta = (x_S \in R\colon S \notin \Delta) \text{ and } I_\Delta = (e_S \in E\colon S \notin \Delta)$$

are the *symmetric* and *exterior Stanley-Reisner ideals of* $\Delta$.

**Definition 3.4.** Let $\Delta$ be a simplicial complex on vertices $[n]$ which is not the $n-1$ simplex. The combinatorial Alexander dual, $\Delta^\star$, is by definition the simplicial complex whose simplices are the complements of the nonsimplices of $\Delta$:

$$\Delta^\star = \{\sigma \subseteq [n]\colon [n] - \sigma \notin \Delta\},$$

It is straightforward to check that, for any $\Delta$, we have

$$\operatorname{ann} I_\Delta = I_{\Delta^\star}. \tag{3.6}$$

For a fixed choice of arrangement and an order on its hyperplanes, the sets

$$\mathbf{nbc} = \{S \subset [n]\colon S \text{ does not contain a broken circuit}\}$$

form a simplicial complex called the *broken circuit complex* (see [6].) From the discussion above, we see $\mathrm{in}(I) = I_{\mathbf{nbc}}$.

**Lemma 3.5.** *For any homogeneous ideal $I$ in $E$, we have* $\mathrm{in}(\operatorname{ann}(I)) = \operatorname{ann}(\mathrm{in}(I))$.

*Proof.* If $x \in \operatorname{ann}(I)$ is a homogeneous element, let $e_S$ be its initial monomial. Replacing $x$ by a nonzero scalar multiple, we get $x = e_S + x'$, where $x'$ is supported on monomials $e_{S'}$ for which $e_S > e_{S'}$ in the term order. Then for any homogeneous $y \in I$, we may similarly write a multiple of $y$ as $y = e_T + y'$, where $e_T$ is the leading term of $y$. Since $e_S e_T$ is the leading monomial in $xy$, it must in fact be zero, since $xy = 0$. This shows $\operatorname{in}(\operatorname{ann}(I)) \subseteq \operatorname{ann}(\operatorname{in}(I))$.

Equality is established by checking that the Hilbert series of the two sides agree:

$$\begin{aligned} H(\operatorname{in}(\operatorname{ann}(I)), t) &= H(\operatorname{ann}(I), t) \\ &= (1+t)^n - t^n H(I, t^{-1}) \text{ by } (3.5); \\ &= (1+t)^n - t^n H(\operatorname{in}(I), t^{-1}) \\ &= H(\operatorname{ann}(\operatorname{in}(I)), t) \end{aligned}$$

□

**Theorem 3.6** ([18])**.** *For any arrangement, $A^*$ has a linear, minimal free resolution*

$$0 \longleftarrow A^* \longleftarrow E(\ell)^{b_{0,-\ell}} \longleftarrow E(\ell-1)^{b_{1,1-\ell}} \cdots \longleftarrow E(\ell-k)^{b_{k,k-\ell}} \longleftarrow \cdots$$

*Sketch of proof:* The first step is to reduce the problem to a one of monomial ideals. Let us ignore degree shifts and replace $A^*$ by $\operatorname{ann} I$ (using (3.5)). By Lemma 3.5 and (3.6), $\operatorname{in}(\operatorname{ann} I) = I_{\mathbf{nbc}^\star}$, since $\operatorname{in}(I) = I_{\mathbf{nbc}}$. If we could show that the monomial ideal $I_{\mathbf{nbc}^\star}$ has a linear resolution, then we would be done: the Gröbner deformation to the initial ideal is upper semicontinuous, so in particular if the initial ideal of $\operatorname{ann}(I)$ has a linear resolution, then so does $\operatorname{ann}(I)$.

It follows from a result of Aramova, Avramov, and Herzog [2] that a monomial ideal $I_\Delta$ in the exterior algebra has a linear resolution if and only if the corresponding squarefree monomial ideal $J_\Delta$ in the polynomial algebra $R$ has a linear resolution. Such ideals have been studied extensively; in particular, Eagon and Reiner [15] show that $J_\Delta$ has a linear resolution if and only if the Stanley-Reisner ideal $J_{\Delta^\star}$ of the Alexander dual complex is Cohen-Macaulay.

Since Alexander duality is an involution, it is enough to know that the ideal $J_{\mathbf{nbc}}$ is Cohen-Macaulay. This amounts to a combinatorial condition on the simplicial complex; see [15] for details. The broken-circuit complex **nbc** is known to be *shellable* [38], a classical combinatorial property which implies $J_{\mathbf{nbc}}$ is Cohen-Macaulay. □

Then the Betti numbers of the resolution are given by (1.3):

$$\begin{aligned} \sum_{i\geq 0} b_{i,i-\ell} t^i &= (-t)^\ell H(A^*, -t)/H(E, -t) \\ &= (-t)^\ell \pi(\mathcal{A}, (-t)^{-1})/(1-t)^n \\ &= \chi(\mathcal{A}, t)/(1-t)^n, \end{aligned}$$

where $\chi(\mathcal{A}, t)$ denotes the characteristic polynomial of the arrangement.

### 3.4. Koszul modules

The definition of the quadratic dual of an algebra (1.8) admits a generalization to modules. Suppose $B$ is a nonnegatively graded $\mathbb{Q}$-algebra and $M$ is a finitely-generated left $B$-module, generated in degree 0. Once again, let $n = \dim_{\mathbb{Q}} B_1$, let $V = B_1$, and suppose further that $M$ has a linear presentation $M = \operatorname{coker} f\colon B^k \to B^m$. Since $f$ is given by a matrix with entries in $B_1$, it is determined by its degree-zero part, which is just a map of vector spaces $f_0\colon \mathbb{Q}^k \to V \otimes \mathbb{Q}^m$.

Then we define $M^!_B$ to be the left $B^!$-module given by the following linear presentation. Let $f_0^\perp\colon \mathbb{Q}^{mn-k} \to V^* \otimes \mathbb{Q}^m$ be a $\mathbb{Q}$-linear map onto the complement of the image of $f_0$. Define a map $f^!\colon (B^!)^{mn-k} \to (B^!)^m$ by letting it act in degree zero by $f_0^\perp$, and extending by the left action of $B^!$. Set

$$M^!_B = \operatorname{coker} f^!\colon (B^!)^{mn-k} \to (B^!)^m. \tag{3.7}$$

As in the case of algebras, the quadratic dual of a module can be understood in terms of resolutions. Let $B$ be a Koszul algebra. If $M$ is a left $B$-module, then the Yoneda product makes $\operatorname{Ext}_B(M, \mathbb{Q})$ a left $B^!$-module, since $B^! = \operatorname{Ext}_B(\mathbb{Q}, \mathbb{Q})$. If $\{b_{ij}\}$ denote as usual the Betti numbers in the minimal free resolution of $M$, then

$$b_{ij} = \dim_{\mathbb{Q}} \operatorname{Ext}^i_B(M, \mathbb{Q})_{-j},$$

for all $i, j \geq 0$. The submodule of $\operatorname{Ext}_B(M, \mathbb{Q})$ corresponding to the linear strand turns out to be isomorphic to the quadratic dual of $M$: that is,

$$\operatorname{Ext}_B(M, \mathbb{Q})_{(0)} \cong M^!_B$$

as left $B^!$-modules. If the module $M$ has a linear free resolution, we have:

$$\operatorname{Ext}_B(M, \mathbb{Q}) = \operatorname{Ext}_B(M, \mathbb{Q})_{(0)} \cong M^!_B.$$

In this case, $M$ is called a *Koszul module*. It is not hard to check that quadratic duality for modules is an involution, which has the following good consequence.

**Proposition 3.7** ([37]). *If $B$ is a Koszul algebra, then $M$ is a Koszul $B$-module if and only if $M^!_B$ is a Koszul $B^!$-module.*

That is, if $M$ has a linear resolution, so does $M^!_B$. Additively, for $p \geq 0$, the $p$th term in the resolution of $M^!_B$ is $(M^*)_{-p} \otimes_{\mathbb{Q}} B^!(-p)$. In this language, Theorem 3.6 says that, for any arrangement, $A^*(-\ell)$ is a Koszul $E$-module. Let $F(\mathcal{A}) = \operatorname{Ext}_S(A^*(-\ell), \mathbb{Q})$, its quadratic dual $R$-module. By Proposition above, $F(\mathcal{A})$ is a Koszul $R$-module. Since $F(\mathcal{A})^!_S \cong A^*(-\ell)$, the module $F(\mathcal{A})$ has a linear resolution of the form

$$0 \longleftarrow F(\mathcal{A}) \longleftarrow A_\ell \otimes S \longleftarrow A_{\ell-1} \otimes S(-1) \cdots \longleftarrow A_0 \otimes S(-\ell) \longleftarrow 0 \tag{3.8}$$

via the identification $A^*(-\ell)^*_{-p} \cong A_{\ell-p}$. It turns out that the differential is given by

$$e \otimes x \mapsto \sum_{i=1}^n ee_i \otimes x_i x,$$

where $\{x_i\}$ is the dual basis to the basis $\{e_i\}$ of $A_1$.

We note that the module $F(\mathcal{A})$ and its resolution (3.8) are originally introduced in [18] by means of Bernstein-Gelfand-Gelfand duality, as developed in Eisenbud-Fløystad-Schreyer [17] and explained in [16, Chapter 7]. BGG duality makes use of the special properties of the Koszul dual rings $E$ and $S$; in particular, since $F(\mathcal{A})$ has a linear resolution, it implies the following:

**Proposition 3.8.** *For all $j$, there is an isomorphism of graded $S$-modules*

$$\operatorname{Ext}_S^{\ell-j}(F(\mathcal{A}), S) \cong \operatorname{Ext}_E(A, \mathbb{Q})_{(j)}.$$

In particular, for each fixed $j$, the Betti numbers $b_{i,i+j}$ in the resolution of $A$ can be interpreted as the Hilbert series of an $S$-module, $\operatorname{Ext}_S^{\ell-j}(F(\mathcal{A}), S)$. With this in mind, Schenck and Suciu [42] proved Theorem 3.1 by understanding the $S$-module $\operatorname{Ext}_S^{\ell-1}(F(\mathcal{A}), S)$ as the linearized Alexander invariant (see [29], and more recently, [32, 33].)

### 3.5. Resonance

In Falk's lecture notes in this volume [19], he defines the resonance varieties of an arrangement, for $p \geq 1$, as

$$\mathcal{R}^p(\mathcal{A}) = \left\{a \in \overline{A}_1 - \{0\} : H^p(\overline{A}, a) \neq 0\right\}, \tag{3.9}$$

where $(\overline{A}, a)$ denotes the projective Orlik-Solomon algebra of $\mathcal{A}$ over $\mathbb{C}$, regarded as a cochain complex with a differential given by (right) multiplication by the element $a$. Since each $\mathcal{R}^p(\mathcal{A})$ is invariant under multiplication by $\mathbb{C}^*$, the orbits form a projective variety $\mathbb{P}\mathcal{R}^p(\mathcal{A})$ in $\mathbb{P}^{n-2}$.

We need to make a slight translation to fit the notation here. Recall from [19] that the derivation $\partial\colon E \to E$ defined by $\partial(e_i) = 1$ induces a well-defined differential on the Orlik-Solomon algebra; moreover, the chain complex $(A_\bullet, \partial)$ is exact, and one may identify $\overline{A}$ with $\ker\partial$ (or, equivalently, $\operatorname{im}(\partial)$.) A more general and geometric discussion may be found in [13].

If $a = \sum_{i=1}^n a_i e_i$, then $\partial(a) = \sum_{i=1}^n a_i$, so $\overline{A}_1 = \{a \in A_1 \colon \sum_{i=1}^n a_i = 0\}$. It is straightforward to check that the map $(\partial a + a\partial)\colon A \to A$ is multiplication by $\sum_{i=1}^n a_i$. Reading this as a chain homotopy on the cochain complex $(A, a)$, multiplication by $\sum_{i=1}^n a_i$ is an isomorphism (over $\mathbb{C}$) unless this sum is zero, so $(A, a)$ is exact unless $a \in \overline{A}_1$. On the other hand, it also says $\partial$ is a chain map provided $a \in \overline{A}_1$ which, together with exactness of $\partial$, gives a short exact sequence of chain complexes

$$0 \longrightarrow (\overline{A}, a) \longrightarrow (A, a) \xrightarrow{\partial} (\overline{A}, a)[-1] \longrightarrow 0.$$

Multiplication by any $a' \in A_1$ with $\partial(a') \neq 0$ gives a section, so the long exact sequence breaks up, and

$$H^p(A, a) = \begin{cases} H^p(\overline{A}, a) \oplus H^{p-1}(\overline{A}, a) & \text{for } a \in \overline{A}_1; \\ 0 & \text{otherwise} \end{cases}$$

for all $p$.

This means that we can look at resonance varieties using the resolution (3.8). The differential in the complex (3.8) is multiplication by the element $\omega = \sum_{i=1}^n e_i \otimes x_i \in A_1 \otimes S_1$, which we can specialize to the differential in any complex $(A, a)$. Formally, for $a \in A_1$, define a homomorphism $S \to \mathbb{Q}$ by $x_i \mapsto a_i$ for $1 \le i \le n$. Let $\mathbb{Q}_a$ denote $\mathbb{Q}$, regarded as a $S$-module in this way. Then, as a consequence of Theorem 3.6,

**Proposition 3.9.** *For all* $0 \le i \le \ell$,
$$H^p(A, a) = \operatorname{Tor}^S_{\ell-p}(F(\mathcal{A}), \mathbb{Q}_a).$$

There is also a projective version of the module $F(\mathcal{A})$: we simply let $\overline{S} = S/(\sum_{i=1}^n x_i)$, and define
$$F(\overline{\mathcal{A}}) = F(\mathcal{A}) \otimes_S \overline{S}.$$
Then, by a modification of the same argument, $F(\overline{\mathcal{A}})$ has a linear resolution over $\overline{S}$:
$$0 \longleftarrow F(\overline{\mathcal{A}}) \longleftarrow \overline{A}_{\ell-1} \otimes \overline{S} \longleftarrow \overline{A}_{\ell-2} \otimes \overline{S}(-1) \cdots \longleftarrow \overline{A}_0 \otimes \overline{S}(-\ell+1) \longleftarrow 0 \tag{3.10}$$
with differential given by multiplying by the image of $\omega$ in $\overline{A} \otimes \overline{S}$. As long as $\partial(a) = 0$, the action of $R$ on $\mathbb{Q}_a$ factors through $\overline{S}$, and we obtain
$$H^p(\overline{A}, a) = \operatorname{Tor}^{\overline{S}}_{\ell-1-p}(F(\overline{A}), \mathbb{Q}_a).$$
Since nonvanishing Tor groups appear with consecutive homological indices, we arrive at another corollary to Theorem 3.6, stated as [18, Theorem 4.1(ii)]:

**Corollary 3.10.** *For any arrangement $\mathcal{A}$ of rank $\ell$, we have*
$$\emptyset = \mathcal{R}^0(\mathcal{A}) \subseteq \mathcal{R}^1(\mathcal{A}) \subseteq \mathcal{R}^2(\mathcal{A}) \subseteq \cdots \subseteq \mathcal{R}^{\ell-1}(\mathcal{A}) = \overline{A}_1.$$

As explained in [14], resonance is closely related to cohomology of local systems. This leads to a question:

*Problem* 3.11. If $\mathcal{L}_\lambda$ is a rank-one local system on the complement $M$ of a rank $\ell$ hyperplane arrangement, is it necessarily the case that
$$H^i(M, \mathcal{L}_\lambda) \neq 0 \Rightarrow H^{i+1}(M, \mathcal{L}_\lambda) \neq 0$$
for all $i < \ell$? (From [14, Corollary 6.7], this follows from Corollary 3.10 for $\lambda$ close to the trivial representation $\mathbb{1}$.)

### 3.6. Resonance and Betti numbers

One can obtain the first resonance variety directly from the module $F(\mathcal{A})$: Schenck and Suciu [42] show that
$$\mathcal{R}^1(\mathcal{A}) \cup \{0\} = V(\operatorname{ann} \operatorname{Ext}^{\ell-1}_S(F(\mathcal{A}), S)), \tag{3.11}$$
the subvariety of $\mathbb{C}^n$ defined by the vanishing of the annihilator ideal. A more complicated relationship occurs for the higher resonance varieties: see [11].

We already saw that the components of the resonance varieties are linear [19, Theorem 4.16], and the components of $\mathbb{P}\mathcal{R}^1(\mathcal{A})$ have empty intersections. Let $h_r$

denotes the number of components of $\mathbb{P}\mathcal{R}^1(\mathcal{A})$ of (projective) dimension $r$, for $r \geq 0$. Using (3.11), these numbers bound the Hilbert series of $\mathrm{Ext}_S^{\ell-1}(F(\mathcal{A}), S)$, which is equivalent to the Betti numbers $\mathrm{Ext}_E^i(A, \mathbb{Q})_{(1)}$ (Proposition 3.8), and equivalent to the Chen ranks (Theorem 3.1). In the last formulation,

$$\theta_k \geq (k-1) \sum_{r \geq 1} h_r \binom{r+k-1}{k} \tag{3.12}$$

for sufficiently large $k$.

The Chen Ranks Conjecture ([44, 42]) states that (3.12) is in fact an equality for all sufficiently large $k$. It has been verified for graphic and decomposable arrangements, by comparing the explicit calculations of the Chen ranks in (3.2) and (3.1), respectively, with matching calculations of the resonance varieties. However, the general case remains open and makes a good problem with which to conclude these lecture notes:

*Problem* 3.12. Show that, for any arrangement $\mathcal{A}$,

$$\theta_k = (k-1) \sum_{r \geq 1} h_r \binom{r+k-1}{k} \qquad \text{for } k >> 0,$$

where $h_r$ is the number of components of $\mathcal{P}\mathcal{R}^1(\mathcal{A})$ of dimension $r$, and $\theta_k$ is the rank of the $k$th lower central series quotient of $G/G''$.

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