A NOTE ON DE CONCINI AND PROCESI'S CURIOUS IDENTITY

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Abstract. We give a short, case-free and combinatorial proof of de Concini and Procesi's formula from [1] for the volume of the simplicial cone spanned by the simple roots of any finite root system. The argument presented here also extends their formula to include the non-crystallographic root systems.

1. Introduction

Let $\Phi \subseteq \mathbb{R}^n$ be a finite root system with base Δ, and let $W = W(\Phi)$ denote the reflection group of Φ. Let σ_Δ be the positive cone spanned by the set of simple roots Δ:

$$\sigma_\Delta = \left\{ \sum_{\alpha \in \Delta} c_\alpha \alpha \colon c_\alpha \in \mathbb{R}_{>0} \text{ for all } \alpha \in \Delta \right\}. \tag{1}$$

Let C_Δ be the normal cone to σ_Δ: this is usually called the fundamental chamber in the arrangement $\mathcal{A}$ of reflecting hyperplanes of W. If τ is a cone in $\mathbb{R}^n$, define the volume of τ as $\nu(\tau) = \mathrm{vol}(\tau \cap D^n)/\mathrm{vol}\ D^n$, where D^n is the unit ball centered at the origin. Finally, let $\{d_1, d_2, \ldots, d_n\}$ denote the degrees of W: we refer to [2] for background and notation.

Recall that the action of W on $\mathbb{R}^n$ by reflections is free on the complement of the hyperplanes $\mathcal{A}$. The induced action on chambers is simply transitive. Since the chambers partition the complement of $\mathcal{A}$ and W acts by isometries, $\nu(gC_\Delta) = 1/|W| = 1/\prod_{i=1}^n d_i$, for any chamber gC_Δ.

While not so straightforward, it turns out that the volume of the cone σ_Δ is also rational, and has a nice expression:

Theorem 1 (Theorem 1.3 in [1]). *If Φ is crystallographic, the volume of the cone σ_Δ is*

$$\nu(\sigma_\Delta) = \prod_{i=1}^n \frac{d_i - 1}{d_i}. \tag{2}$$

De Concini and Procesi derive this result from the "curious identity" of their title. Their proof of the identity is accompanied by a note by Stembridge that gives an elegant, alternate proof via character theory.

The purpose of this note is to offer yet another argument. Using the combinatorial theory of real hyperplane arrangements, one can prove (2) directly, in slightly more generality (§2). Then, in the crystallographic case, de Concini and Procesi's identity is recovered by adding up normal cones around the fundamental alcove of the associated affine root system $\widetilde{\Phi}$ (in §3).

2. The volume formula

Let $V \subseteq \mathbb{R}^n$ consist of the union of the reflecting hyperplanes, together with those vectors in the span of any proper subset of any base $g\Delta$. Clearly $\mathbb{R}^n - V$ is a dense, open subset of $\mathbb{R}^n$. The key result is the following, whose proof appears at the end of this section.

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Theorem 2. *For any $x \in \mathbb{R}^n - V$, the number of $g \in W$ for which $x \in g\sigma_\Delta$ is independent of x and equal to $\prod_{i=1}^n (d_i - 1)$.*

In another formulation,

Corollary 3. *For a finite root system Φ and $x \in \mathbb{R}^n - V$, the number of choices of base Δ for Φ for which x is in the positive cone of Δ equals $\prod_{i=1}^n (d_i - 1)$.*

Proof. If Δ, Δ' are both bases for Φ, then $\Delta' = g\Delta$ for some $g \in W$, and $\sigma_{\Delta'} = g\sigma_\Delta$. □

Since each cone $g\sigma_\Delta$ has the same volume,

$$\begin{aligned} |W| \cdot \nu(\sigma_\Delta) &= \sum_{g \in W} \nu(g\sigma_\Delta) \\ &= \prod_{i=1}^n (d_i - 1) \end{aligned}$$

by Theorem 2, and we obtain the volume formula as a corollary:

Theorem 1$^+$. *If Φ is any finite root system, the volume of the cone σ_Δ is*

$$\nu(\sigma_\Delta) = \prod_{i=1}^n \frac{d_i - 1}{d_i}.$$

(Note that, if the rank of Φ is less than n, the least degree is 1, and both sides are zero.)

2.1. **Hyperplane arrangements.** The terminology used below may be found in the book of Orlik and Terao [3]. We recall a collection of hyperplanes $\mathcal{A}$ in $\mathbb{R}^n$ is *central* if all $H \in \mathcal{A}$ contain the origin, and *essential* if the collection of normal vectors span $\mathbb{R}^n$.

Recall that $\mathcal{A}$ has an intersection lattice $L(\mathcal{A})$ of subspaces, ranked by codimension. The Poincaré polynomial of $\mathcal{A}$ is defined to be

$$\pi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(\widehat{0}, X)(-t)^{\operatorname{rank}(X)},$$

where μ is the Möbius function. If $\mathcal{A}$ is essential, $\pi(\mathcal{A}, t)$ is a polynomial of degree n. The following classical theorem is a main ingredient in our proof.

Theorem 4 ([4])**.** *If $\mathcal{A} = \mathcal{A}(\Phi)$ is an arrangement of (real) reflecting hyperplanes, then*

$$\pi(\mathcal{A}, t) = \prod_{i=1}^n (1 + (d_i - 1)t), \tag{3}$$

where $\{d_i\}$ are the degrees of the reflection group.

If H_0 is any hyperplane (not necessarily through the origin), let $\mathcal{A}^{H_0}$ denote the set $\{H \cap H_0 : H \in \mathcal{A}\}$, regarded as a hyperplane arrangement in H_0. We say H_0 is in general position to $\mathcal{A}$ if $X \cap H_0$ is nonempty for all nonzero subspaces $X \in L(\mathcal{A})$.

Lemma 5. *If H_0 is in general position to a central arrangement $\mathcal{A}$ in $\mathbb{R}^n$, then the number of bounded chambers in $\mathcal{A}^{H_0}$ equals the coefficient of t^n in $\pi(\mathcal{A}, t)$.*

Proof. It follows from the definition of general position that $L(\mathcal{A}^{H_0}) = L(\mathcal{A})_{\leq n-1}$, where the latter is the truncation of the lattice $L(\mathcal{A})$ to rank $n - 1$. Therefore $\pi(\mathcal{A}, t) = \pi(\mathcal{A}^{H_0}, t) + bt^n$ for some b. By a theorem of Zaslavsky [6], the number of bounded chambers of any arrangement $\mathcal{B}$ equals $(-1)^{\operatorname{rank} \mathcal{B}} \pi(\mathcal{B}, -1)$. Substituting $t = -1$ shows b is the number of bounded chambers in $\mathcal{A}^{H_0}$, since $\mathcal{A}$ itself has none. □

Let $\epsilon > 0$ be a fixed choice of positive, real number.

Lemma 6. *For any $x \in C_\Delta \cap (\mathbb{R}^n - V)$ let H_x be the hyperplane normal to x, passing through ϵx. Then H_x is in general position to $\mathcal{A}$.*

Proof. Suppose $X \cap H_x = \emptyset$ for some nonzero intersection of hyperplanes X. Say $X = \cap_{\alpha \in S} H_\alpha$, where $S \subseteq \Phi$. Since $X \neq 0$, the roots S do not span $\mathbb{R}^n$. Since X and H_x are parallel, x is a linear combination of the roots S; then $x \in V$, a contradiction. $\square$

For each $y \in \mathbb{R}^n$ with $(x, y) > 0$, let y^{H_x} denote the unique, positive multiple of y which lies in H_x. Note that each chamber of $\mathcal{A}^{H_x}$ has the form $C \cap H_x$ for some chamber C of $\mathcal{A}$. If $C \cap H_x$ is bounded, then C is just a cone over $C \cap H_x$ with retraction $y \mapsto y^{H_x}$. In particular, $(x, y) > 0$ for all $y \in C$. For any $x \in \mathbb{R}^n - V$, let

$$B_x = \left\{ g \in W : (x, gx) > 0 \text{ and } (gx)^{H_x} \text{ is in a bounded chamber of } \mathcal{A}^{H_x} \right\}. \tag{4}$$

Since $x \notin V$, the orbit Wx has exactly one point in each chamber of $\mathcal{A}$. It follows that $|B_x|$ is the number of bounded chambers of $\mathcal{A}^{H_x}$.

Lemma 7. *For any $x \in \mathbb{R}^n - V$, we have*

$$B_x = \left\{ g \in W : g^{-1}x \in \sigma_\Delta \right\}.$$

Proof. A chamber $C \cap H_x$ of $\mathcal{A}^{H_x}$ is bounded if and only if C does not contain a ray in H_x. Equivalently, all points in $C \cap H_x$ (or, just as well, in C) have positive inner product with respect to x.

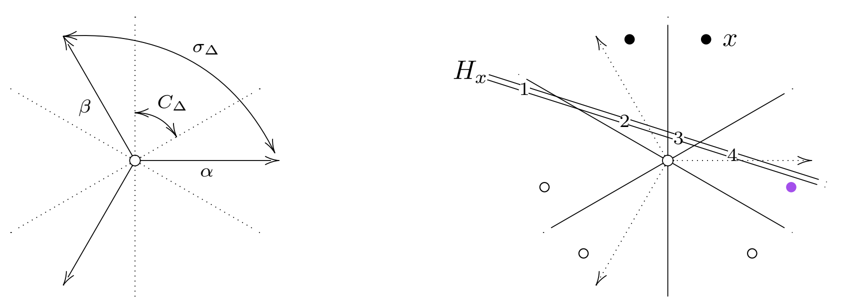
That is, $g \in B_x$ if and only if, for all $y \in C_\Delta$,

$$(gy, x) > 0 \iff (y, g^{-1}x) > 0 \iff g^{-1}x \in \sigma_\Delta,$$

since σ_Δ is the normal cone to C_Δ. $\square$

2.2. **Proof of Theorem 2.** Fix a point $x \in \mathbb{R}^n - V$. By construction, x lies in some (open) chamber C. Without loss of generality, $C = C_\Delta$. Let H_x be the hyperplane normal to x, containing ϵx. Using Lemmas 5, 6, and equation (3), we see the number of bounded chambers in $\mathcal{A}^{H_x}$ equals $\prod_{i=1}^n (d_i - 1)$.

On the other hand, the number of bounded chambers of $\mathcal{A}^{H_x}$ equals $|B_x|$; by Lemma 7, this equals the number of $g \in W$ for which $x \in g\sigma_\Delta$. $\square$



(a) The cone σ_Δ and chamber C_Δ (b) Chambers of $\mathcal{A}^{H_x}$ and the orbit of x

FIGURE 1. The A_2 root system

Example 1. Let $\Delta = \{\alpha, \beta\}$ be the base of the A_2 root system, shown in Figure 1(a). Recall $d_1 = 2, d_2 = 3$; then $\nu(\sigma_\Delta) = \frac{1 \cdot 2}{2 \cdot 3}$. In Figure 1(b), the chambers of $\mathcal{A}^{H_x}$ are labelled 1 through 4. As expected, two chambers (labelled 2 and 3) are bounded. For a given $x \in C_\Delta$, points gx in its orbit are marked with a "$\circ$" if $(x, gx) \leq 0$. If $(x, gx) > 0$, the point gx is black where the chamber $(gx)^{H_x}$ is bounded and "$\bullet$" otherwise.

3. The identity

Now suppose that $\Phi \subseteq \mathbb{R}^n$ is an irreducible, crystallographic root system of rank n. Let $\widetilde{\Phi}$ denote the affine root system of Φ, with base $\widetilde{\Delta} = \Delta \cup \{\alpha_0\}$. Let $\widetilde{D}$ denote the extended Dynkin diagram of Φ. For each simple root $\alpha_i \in \widetilde{\Delta}$, let Φ_i be the sub-root system of Φ with base $\Delta_i = \widetilde{\Delta} - \{\alpha_i\}$. Then $\Phi = \Phi_0$, and recall that the Dynkin diagram of Φ_i is obtained by deleting the vertex corresponding to α_i from $\widetilde{D}$.

For each i, $0 \leq i \leq n$, let $(d_1^{(i)}, \ldots, d_n^{(i)})$ denote the degrees of Φ_i. De Concini and Procesi found that, for each irreducible type, an unexpected identity held:

Theorem 8 (Theorem 1.2 of [1])**.** *For an irreducible, crystallographic root system Φ of rank n,*

$$\sum_{i=0}^{n} \prod_{j=1}^{n} \frac{d_j^{(i)} - 1}{d_j^{(i)}} = 1. \tag{5}$$

By (re)deriving their result from Theorem 1, a geometric interpretation becomes apparent.

Proof. Let A_0 denote the fundamental alcove of Φ. This is a simplex bounded by the (affine) reflecting hyperplanes $\{H_{\alpha_i} \colon 0 \leq i \leq n\}$. For each i, let v_i be the vertex of A_0 that is opposite the face contained in H_{α_i}. The normal cone to A_0 at v_i is spanned by the vectors $\widetilde{\Delta} - \{\alpha_i\}$, so it is just the cone σ_{Δ_i}. Then

$$\nu(\sigma_{\Delta_i}) = \prod_{j=1}^{n} \frac{d_j^{(i)} - 1}{d_j^{(i)}},$$

by the volume formula (2). However, the normal cones to the vertices of any polytope partition a dense open subset of $\mathbb{R}^n$, so their volumes sum to 1. $\square$

Remark 1. We have seen that the volume formula (2) also holds for finite, noncrystallographic root systems. For the irreducible types, (2) gives

Type	$I_2(m)$	H_3	H_4
$\nu(\sigma_\Delta)$	$(m-1)/(2m)$	$3/8$	$6061/14\,400$

Although the identity (5) no longer makes sense, one might still be tempted to compute the left side formally for diagrams that extend H_3 or H_4 by a vertex in such a way that all proper subdiagrams are of finite type. (These include the Coxeter groups H_3^{aff} and H_4^{aff} of Patera and Twarock, [5].) Perhaps unsurprisingly, however, an exhaustive search shows that the identity fails to hold for any such diagram.

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