Equations for loci of commuting nilpotent matrices

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Abstract

The Jordan type of a nilpotent matrix A is the partition P_A giving the sizes of the Jordan blocks of the Jordan matrix in its conjugacy class. For $Q = (u, u - r)$ with r at least 2, there is a known table $\mathcal{T}(Q)$ of Jordan types P for $n \times n$ matrices whose maximum commuting nilpotent Jordan type $\mathfrak{Q}(P)$ is Q (arXiv math 1409.2192). Let B be the Jordan matrix of partition Q, and consider the affine space $\mathcal{N}_B$ parametrizing nilpotent matrices commuting with B. For a partition P in $\mathcal{T}(Q)$, the locus $\mathfrak{Z}(P)$ of P is the subvariety of $\mathcal{N}_B$ parametrizing matrices A having Jordan type P. In this talk we outline conjectures and results concerning the equations defining $\mathfrak{Z}(P)$. If time permits, we state analogous loci equation conjectures for partitions in the boxes $\mathcal{B}(Q)$ when Q has three or more parts.

Section 0: The map $\mathfrak{Q} : P \to \mathfrak{Q}(P)$

Definition (Nilpotent commutator $\mathcal{N}_B$ and $\mathfrak{Q}(P)$.)

$V \cong k^n$ vector space over an infinite field k.
$A, B \in \mathrm{Mat}_n(k) = \mathrm{Hom}_k(V, V)$, nilpotent matrices.

$\mathcal{C}_B \subset \mathrm{Mat}_n(k)$ centralizer of B.

$\mathcal{N}_B \subset \mathcal{C}_B$: the variety of nilpotent elements of $\mathcal{C}_B$.

$P \vdash n$ is a partition of n;
J_P = Jordan block matrix, the sizes of whose blocks is P.

P_A = Jordan type of A – the partition such that $J_{P_A} = CAC^{-1}$ is similar to A.

$\mathfrak{Q}(P)$: the maximum Jordan type in Bruhat order of a nilpotent matrix commuting with J_P.

$r_P = \#$ almost rectangular partitions (parts differ at most by 1) needed to cover P.

Problem 1: Determine the map $\mathfrak{Q} : P \rightarrow \mathfrak{Q}(P)$.

Fact (T. Košir, P. Oblak): $\mathfrak{Q}(P)$ is *stable*: parts differ pairwise by at least 2 and $\mathfrak{Q}(\mathfrak{Q}(P)) = \mathfrak{Q}(P)$.

Fact (R. Basili): $\mathfrak{Q}(P)$ has r_P parts.

Partial Answers: Oblak Recursive Conjecture:
$\mathfrak{Q}(P) = Oblak(P)$. Known for $Q = \mathfrak{Q}(P)$ with 2 or 3 parts
(P. Oblak, L. Khatami).
Thm: $Oblak(P) = \lambda_U(P) \leq \mathfrak{Q}(P)$ (L. Khatami, L.K. and A.I.).

Problem 2: Given Q determine all P such that $\mathfrak{Q}(P) = Q$.

Partial Answer: i. **Table Theorem** for $Q = (u, u - r), r \geq 2$
(A.I., L.Khatami, B.Van Steirtegham, R. Zhao).

ii. **Equations conjecture** and **Box Conjecture**.

Claim: **These should have been classical problems!** Canonical form is due to C. Jordan, 1870. But the map $P \to \mathfrak{Q}(P)$ was not studied classically.[1]

In 2006, three independent groups began to work on the $P \to \mathfrak{Q}(P)$ problem
    P. Oblak and T. Košir (Ljubljana)
    D. Panyushev (Moscow)
    R. Basili, I.-, and L. Khatami (Perugia, Boston).

Connected to Hilbert scheme work of J. Briançon, M. Granger, R. Basili, V. Baranovsky, A. Premet, N. Ngo and K. Šivic.

Links to work of E. Friedlander, J. Pevtsova, A. Suslin, on representations of Abelian p-groups [FrPS,CFrP].

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[1]Instead, I. Schur (1905), N.Jordan, M. Gerstenhaber (1958), E. Wang (1979) studied *maximum dimension* commuting subalgebras/nilpotent subalgebras of matrices.

Section 1: Artinian Gorenstein quotients of $R = \mathsf{k}\{x, y\}$ and Jordan type of multiplication maps.

When the Hilbert function H of an Artinian R- module $\mathfrak{X}$ is fixed, the conjugate partition $H^{\vee}$ is an upper bound for the partitions that might occur as the Jordan type P_x for the multiplication m_x on $\mathfrak{X}$ by $x \in R$. Given H what are the possible Jordan types $P_y, y \in R$ for m_y on $\mathfrak{X}$? Conversely, let $P = P_A$: what is the maximum Jordan type $\mathfrak{Q}(P)$ in Bruhat order of a nilpotent matrix B commuting with A?

Example

Let $\mathcal{A} = \mathsf{k}\{x, y\}/I, I = (xy, y^2 + x^3) = f^{\perp}$ where $f = Y^2 - X^3 \in \mathsf{k}_{DP}[X, Y]$. Here $H(\mathcal{A}) = (1, 2, 1, 1)$ and as $\mathsf{k}[x]$ module $\mathcal{A} \cong \langle 1, x, x^2, x^3; y \rangle$, so $P_x = (4, 1) = H^{\vee}$.

Question. What are the possible Jordan types P_A of $m_A, A \in \mathcal{A}$?

Variation Fix $Q = (4, 1)$. Assume Q is the *maximum* Jordan type $Q = \mathfrak{Q}(P)$ (in Bruhat order) of a nilpotent matrix B commuting with a matrix A. What are the possible Jordan types $P = P_A$?

Answer Besides $(4, 1)$, here $P = (3, 1, 1)$ is the only other partition for which $\mathfrak{Q}(P) = (4, 1)$.

We say $Q = (u, u - r)$ is *stable* if $u > r \geq 2$ (i.e. if its parts differ pairwise by at least 2). The last four authors show the following in [IKVZ].

Theorem (Table theorem)

Let $Q = (u, u - r)$ be stable. Then there are exactly $(r - 1)(u - r)$ partitions $P_{ij}(Q), 1 \leq i \leq r - 1, 1 \leq j \leq u - r$ such that $\mathfrak{Q}(P_{ij}) = Q$. These form a table $\mathcal{T}(Q)$ and P_{ij} has $i + j$ parts. The table is comprised of B hooks and A rows or partial rows that fit together as in a puzzle.

An AR (almost rectangular) partition has parts differing pairwise by at most 1. Notation: $[n]^k$ is the AR partition of n into k parts.

Example

Let $Q = (8, 3)$. Then

$$\mathcal{T}(8,3) = \begin{pmatrix} (8,3) & (8,[3]^2) & (8,[3]^3) \\ (\mathbf{5},[\mathbf{6}]^2) & ([8]^2,[3]^2) & ([8]^2,[3]^3) \\ (\mathbf{5},[\mathbf{6}]^3) & ([7]^2,[4]^3) & ([7]^2,[4]^4) \\ (\mathbf{5},[\mathbf{6}]^4) & (\mathbf{5},[\mathbf{6}]^5) & (\mathbf{5},[\mathbf{6}]^6) \end{pmatrix}$$

$$= \begin{pmatrix} (8,3) & (8,2,1) & (8,1^3) \\ (\mathbf{5},\mathbf{3},\mathbf{3}) & (4,4,2,1) & (4,4,1^3) \\ (\mathbf{5},\mathbf{2},\mathbf{2},\mathbf{2}) & (4,3,2,1,1) & (4,3,1^4) \\ (\mathbf{5},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) & (\mathbf{5},\mathbf{2},\mathbf{1}^4) & (\mathbf{5},\mathbf{1}^6) \end{pmatrix}$$

red − first B hook   *blue − second B hook*

Def. The **diagram** of a partition P is a poset whose rows are the parts of P (P. Oblak, L. Khatami)
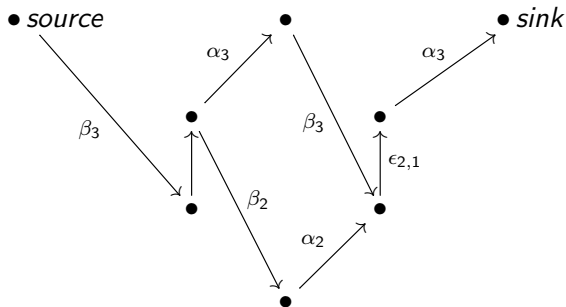
Figure : $\mathrm{Diag}(\mathcal{D}_P)$ for $P = (3, 2, 2, 1)$.

Def: U-chain in $\mathcal{D}_P$ determined by an AR $P_c \subset P$: a chain that includes all vertices of $\mathcal{D}_P$ from an AR subpartition P_c, $+$ two tails.

The first tail descends from the source of $\mathcal{D}_P$ to the AR chain of P_c, and the second tail ascends from the AR chain to the sink of $\mathcal{D}_P$.

Oblak Recursive Conjecture

One obtains $\mathfrak{Q}(P)$ from $\mathcal{D}_P$:

(i) Let C be a longest U-chain of $\mathcal{D}_P$. Then $|C| = q_1$, the biggest part of $\mathfrak{Q}(P)$.

(ii) Remove the vertices of C from $\mathcal{D}_P$, giving a partition $P' = P - C$. If $P' \neq \emptyset$ then $\mathfrak{Q}(P) = (q_1, \mathfrak{Q}(P'))$ (Go to (i).).

Known for Q stable with two or three parts (P. Oblak determines the largest part, and L. Khatami the smallest part of $\mathfrak{Q}(P)$).
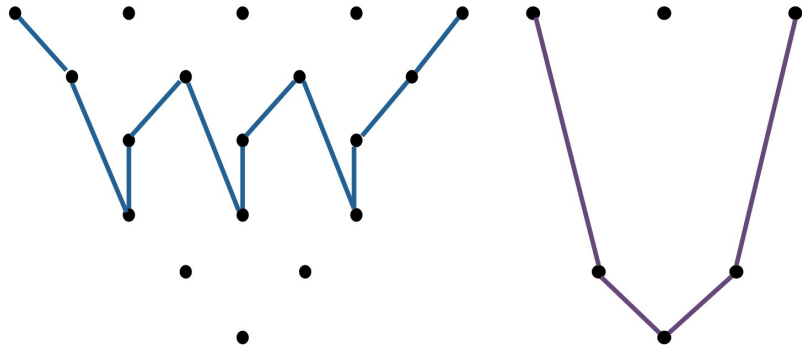
Figure : U-chain C_4: $P = (5, 4, 3, 3, 2, 1)$ and new U-chain of $P' = P - C_4 = (3, 2, 1)$.
$\mathfrak{Q}(P) = (12, 5, 1)$ [figure from LK NU GASC talk 2013]

Section 2: Table Loci

Assume that $Q = (u, u - r)$ is stable. Recall that $B = J_Q$, the nilpotent Jordan block matrix of a partition Q above, and $\mathcal{N}_B =$ family of nilpotent matrices commuting with B.

Def. Let $P_{ij} \in \mathcal{T}(Q)$. Then the *locus* $\mathfrak{Z}(P_{ij})$ is the subvariety of $\mathcal{N}_B$ parametrizing matrices A such that $P_A = P_{ij}(Q)$.

Table Loci Conjecture for stable Q with two parts The locus $\mathfrak{Z}(P_{ij})$ in $\mathcal{N}_B$, is a complete intersection (CI) defined by a specified set of $i + j$ linear and quadratic equations.

Degenerate case, when $Q = (5)$ has a single part
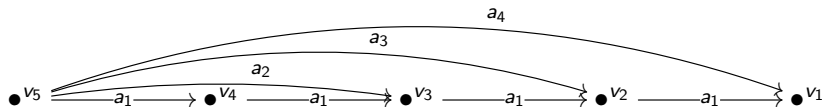
Figure : Diagram of $\mathcal{D}_Q$ and maps for $Q = (5)$.

Example (Diagram and equations of column loci for $Q = (5)$.)

When[2] $a_1 = 0, a_2 \neq 0$ then we have strings (cyclic modules)
$v_5 \rightarrow v_3 \rightarrow v_1$ and $v_4 \rightarrow v_2$ so $P_A = (3, 2) = [5]^2$.

When $a_1 = a_2 = 0, a_3 \neq 0$ then we have strings
$v_5 \rightarrow v_2$ and $v_4 \rightarrow v_1$ and v_3, so $P_A = (2, 2, 1) = [5]^3$.

[2]We write a_1 for x_{a_1}, ...

Table and table equations - single columns - for $Q = (5)$
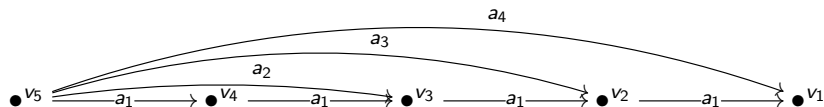
$\mathcal{T}(Q)$ and $\mathcal{E}(Q)$ for $Q = (5)$.
Here $B = J_Q : a_1 = 1, a_2 = a_3 = a_4 = 0$.

$\mathcal{T}(Q)$	$\mathcal{E}(Q)$
(5)	$-$
$[5]^2 = (3, 2)$	a_1
$[5]^3 = (2, 2, 1)$	a_1, a_2
$[5]^4 = (2, 1^3)$	a_1, a_2, a_3
$[5]^5 = (1^5)$	a_1, a_2, a_3, a_4

$A \in \mathcal{N}_B$

$$
\begin{matrix}
0 & a_1 & a_2 & a_3 & a_4 \\
0 & 0 & a_1 & a_2 & a_3 \\
0 & 0 & 0 & a_1 & a_2 \\
0 & 0 & 0 & 0 & a_1 \\
0 & 0 & 0 & 0 & 0
\end{matrix}
$$
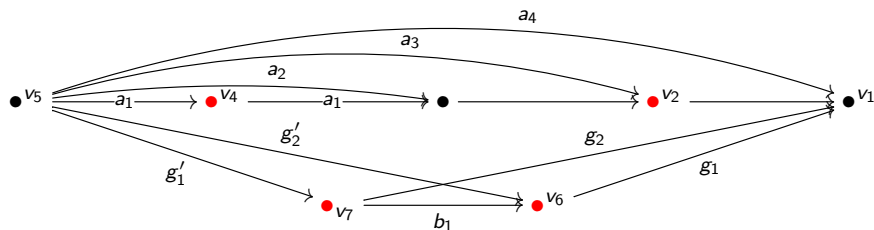
Figure : Diagram of $\mathcal{D}_Q$ and maps for $Q = (5, 2)$.

Example (Equations for table loci: $\mathcal{T}(Q), Q = (5, 2)$)

$$\mathcal{T} = \begin{pmatrix} (5, 2) & (5, [2]^2) \\ (4, [3]^2) & (4, [3]^3) \end{pmatrix}; \qquad \mathcal{E} = \begin{pmatrix} - & b_1 \\ a_1 & a_1, \mathsf{Q} \end{pmatrix}$$

$$\text{where } \mathsf{Q} = \begin{vmatrix} a_2 & g_1 \\ g_1' & b_1 \end{vmatrix}.$$

When $a_1 = 0$, rest general, then $P_A = (4, 2, 1)$.

When $a_1 = 0$ but $Q \neq 0$ then $\ker(A) = \langle v_1, v_2, g_1 v_3 - a_2 v_6 \rangle$.

Equations a_1, Q for $P_{2,2}((5, 2)) = (4, 1, 1, 1)$
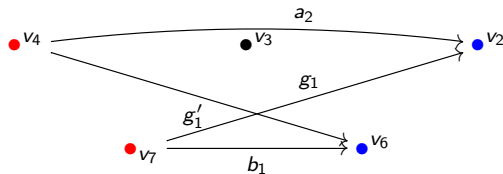
$$Q_{matrix} = \left(\begin{array}{c||cc} * & v_4 & v_7 \\ \hline\hline v_2 & a_2 & g_1 \\ v_6 & g_1' & b_1 \end{array} \right).$$

$\ker(A) = \langle v_1, v_2, g_1 v_3 - a_2 v_6, g_1 v_4 - a_2 v_7 + \cdots \rangle$, so $k_A = 4$.

Example (Table $\mathcal{T}(Q)$ and Table Loci $\mathcal{E}(Q)$ for $Q = (6,3)$)

Let $Q = (6,3)$.

$$\mathcal{T}(Q) = \begin{pmatrix} (6,3) & (6,[3]^2) & (6,[3]^3) \\ (5,[4]^2) & (5,[4]^3) & (5,[4[^4) \end{pmatrix}$$

$$\mathcal{E}(Q) = \begin{pmatrix} - & b_1 & b_1, Q' \\ a_1 & a_1, Q_1 & a_1, Q_1, Q_2 \end{pmatrix}$$

where $Q' = \begin{vmatrix} a_1 & g_1 \\ g_1' & b_2 \end{vmatrix}$, $\qquad Q_1 = \begin{vmatrix} a_2 & g_1 \\ g_1' & b_1 \end{vmatrix}$,

$$Q_2 = \begin{vmatrix} a_2 & g_1 \\ g_2' & b_2 \end{vmatrix} + \begin{vmatrix} a_3 & g_2 \\ g_1' & b_1 \end{vmatrix}$$

The matrix $\mathfrak{M}_A$ over $R = k[t]/(t^u)$ and Diagonalization Loci.

We encode the matrix A for $Q = (u, u - r)$ in the following 2×2 matrix $\mathfrak{M}_A$ over $R = k[t]/(t^u)$, after the Ljubljana school (T. Košir and B. Sethuranam) variation á la M. Boij et al.

$$\mathfrak{M}_A = \left(\begin{array}{cc} a & g \\ ht^r & b \end{array} \right)$$

Here, generically, $a = a_1 t + a_2 t^2 + \cdots , b = b_1 t + b_2 t^2 + \cdots$
$h = h_0 + h_1 t + \cdots ,$ and $g = g_0 + g_1 t + \cdots .$
We have $\dim_k Ker(\mathfrak{M}_A) = k_A$. For a particular choice of coefficients, we may try to row-reduce $\mathfrak{M}_A$ over R to echelon form

$$\mathfrak{M}'_A = \left(\begin{array}{cc} u t^{s_1} & g \\ 0 & u' t^{s_2} \end{array} \right) \text{ where } u, u' \text{ are units.} \qquad (1)$$

Definition

Fix a stable $Q = (u, u - r)$. Let (s_1, s_2) satisfy

$$1 \le s_1 \le r - 1 \text{ and } 1 \le s_2 \le u - r. \tag{2}$$

The *diagonalization locus* $\mathfrak{K}_{s_1,s_2}(Q) = \{A \in \mathcal{N}_B, B = J_Q \text{ s.t. }$
$\mathfrak{M}_A \cong \begin{pmatrix} \mathrm{u}t^{s_1} & g \\ 0 & \mathrm{u}'t^{s_2} \end{pmatrix}\}$ with u, u' units in R.

Lemma

A general enough $A \in \mathfrak{K}_{s_1,s_2}(Q)$ *satisfies* $k_A = s_1 + s_2$, *so* P_A *has* $s_1 + s_2$ *parts.*

Conjecture.

  i For pairs (s_1, s_2) satisfying (2) the rank of each power A^i of a
     general enough $A \in \mathfrak{K}_{s_1,s_2}$ is determined by the pair (s_1, s_2).

  ii The table locus $\mathfrak{Z}(P_{ij})$ is just the diagonalization locus $\mathfrak{X}_{s_1,s_2}$.

We use $\mathfrak{X}_{2,1}$ in a special case to determine $P_{2,1}$.

Example

The locus $\mathfrak{Z}(P_{2,1})$ for $Q = (6,3)$ and $\mathfrak{K}_{2,1}(Q)$.

When $a_1 = 0$ and a_2, b_1, g, h are generic we have

$$\mathfrak{M}_A = \begin{pmatrix} \mathsf{u}t^2 & g \\ ht^3 & \mathsf{u}'t \end{pmatrix} \text{ with } \mathsf{u}, \mathsf{u}' \text{ units. Then we have}$$

$$\mathfrak{M}_A \cong \begin{pmatrix} \mathsf{u}t^2 & g \\ 0 & \mathsf{u}''t \end{pmatrix}, \text{ so this is in the locus } \mathfrak{K}_{2,1}, \text{ where}$$

$k_A = 2 + 1 = 3$. Now consider the dimension k_{A^2} of the kernel of A^2. For $P_{2,1}(Q) = (5,2,2)$ we expect $k_{A^2} = 6$, $k_{A^3} = 7$, $k_{A^4} = 8$ and $A^5 = 0$.

We have $\mathfrak{M}_A^2 = \begin{pmatrix} ght^3 + \mathsf{u}^2 t^4 & g\mathsf{u}'t + g\mathsf{u}t^2 \\ h\mathsf{u}'t^4 + h\mathsf{u}t^5 & \mathsf{u}'^2 t^2 + ght^3 \end{pmatrix}$.

The determinant formed from the lowest order terms is zero, so in row-reducing $\mathfrak{M}_{A^2}$ we obtain orders $\begin{pmatrix} 3 & 1 \\ -\infty & 3 \end{pmatrix}$, giving $k_{A^2} = 6$. It is easy to check $k_{A^3} = 7$, $k_{A^4} = 8$, $A^5 = 0$.

$\mathfrak{D} \mathfrak{Q} \mathfrak{C}$

Set up for the Box equation conjecture.
Let $Q = (q_1, q_2, q_3)$ be a stable partition: so
$q_1 > q_2 + 1, q_2 > q_3 + 1$.
Set $\delta_1 = q_1 - q_2, \delta_2 = q_2 - q_3, \delta_3 = q_3 - (-1)$.
The box entries P_{s_1,s_2,s_3} (whatever they may be) are labelled by the triples (s_1, s_2, s_3) satisfying

$$1 \leq s_1 \leq \delta_1 - 1, 1 \leq s_2 \leq \delta_2 - 1, 1 \leq s_3 \leq \delta_3 - 1. \qquad (3)$$

We denote by $\mathfrak{U}_{s_1,s_2,s_3}(Q)$ the locus in $\mathcal{U}_B, B = J_Q$ of matrices A having Jordan type $P_A = P_{s_1,s_2,s_3}$, and by $\mathfrak{Z}_{s_1,s_2,s_3}(Q)$ its closure in $\mathcal{U}_Q$. Let $R = \mathsf{k}[t]/(t^{q_1})$. The matrix $\mathfrak{M}_A$ is the 3×3 analogue of the 2×2 case, so we define the diagonalization loci $\mathfrak{K}_{s_1,s_2,s_3}$.

We propose

Box Equations Conjecture. Assume that the triple (s_1, s_2, s_3) for $A \in \mathcal{U}_Q$ satisfies (3). Then

i. The locus $\mathfrak{U}_{s_1,s_2,s_3}(Q) = \mathfrak{K}_{s_1,s_2,s_3}(Q)$.

ii. For $A \in \mathfrak{K}_{s_1,s_1,s_3}(Q)$ the triple (s_1, s_2, s_3) determines the rank of each power of A, hence determines a partition $P_A = P\mathfrak{K}_{s_1,s_2,s_3}$.

iii. If $P_A = P\mathfrak{K}_{s_1,s_2,s_3}(Q)$ then $Q(P_A) = Q$.

iv. The set $\{P\mathfrak{K}_{s_1,s_2,s_3}(Q)\}$ such that (s_1, s_2, s_3) satisfies (3) is the complete set of partitions P such that $Q(P) = Q$.

Here (i.) and (ii.) together assert that $P_{s_1,s_2,s_3} = P\mathfrak{K}_{s_1,s_2,s_3}$.
This conjecture is analogous to the Table Conjecture and the Ranks of Powers Conjecture for stable partitions Q having two parts.

Lemma (in process)

The diagonalization loci $\mathfrak{K}_{s_1,s_2}$, and for Q with three parts, $\mathfrak{K}_{s_1,s_2,s_3}$ for permissible triples (s_1, s_2, s_3) are complete intersections defined by specific irreducible equations, whose format and degrees are known.

Conclusion There are many algebraic and combinatorial problems arising from considering two commuting nilpotent matrices!

Thank you!

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