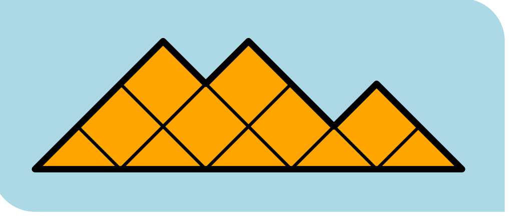
A second proof of the Shareshian–Wachs conjecture

Mathieu Guay-Paquet, LaCIM, Montréal

Sat 23 Jan 2016, CAAC, London

arxiv:1601.05498

The conjecture



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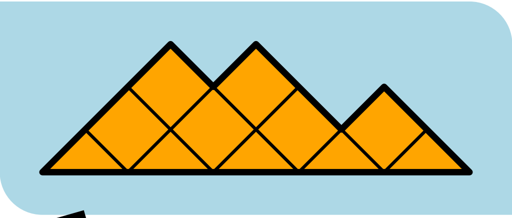
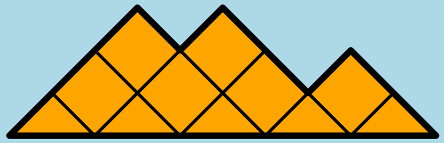


Indifference graph

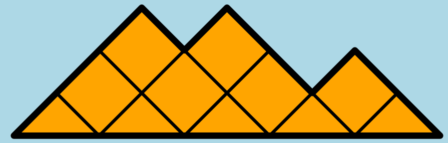
The conjecture



Indifference graph

Count multiplicities and ascents in all proper colourings

The conjecture



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q-chromatic quasisymmetric function

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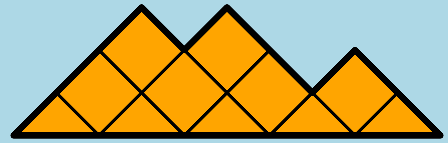

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$$(q^2+2q^3+q^4)m_{221} + (q+6q^2+10q^3+6q^4+q^5)m_{2111} + (1+8q+29q^2+44q^3+29q^4+8q^5+q^6)m_{11111}$$

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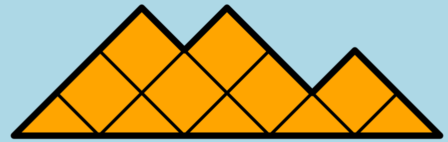

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Hessenberg subvariety of $\mathrm{Flag}(\mathbb{C}^5)$

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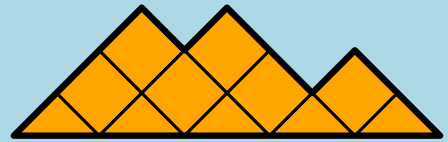

Indifference graph → Count multiplicities and ascents in all proper colourings → q-chromatic quasisymmetric function

Hessenberg subvariety of $\mathrm{Flag}(\mathbb{C}^5)$ → Equivariant cohomology

$$(q^2 + 2q^3 + q^4)m_{221} + (q + 6q^2 + 10q^3 + 6q^4 + q^5)m_{2111} + (1 + 8q + 29q^2 + 44q^3 + 29q^4 + 8q^5 + q^6)m_{11111}$$

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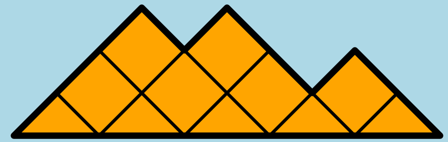

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Equivariant cohomology

Action of S_5

$$(q^2 + 2q^3 + q^4)m_{221}$$
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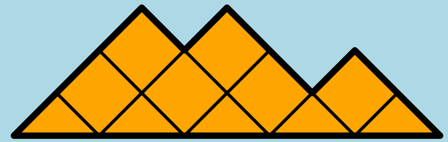

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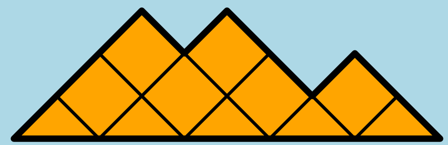

Indifference graph → Count multiplicities and ascents in all proper colourings → q-chromatic quasisymmetric function

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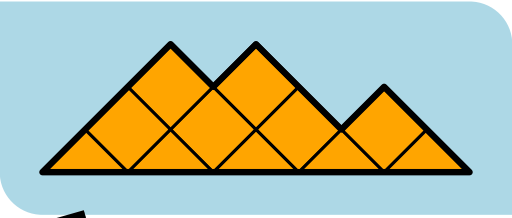
How it works: graphs



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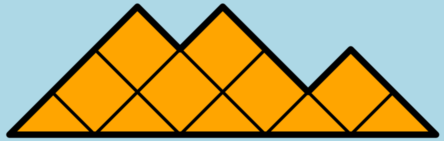

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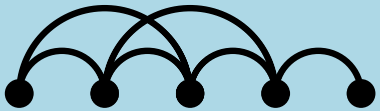

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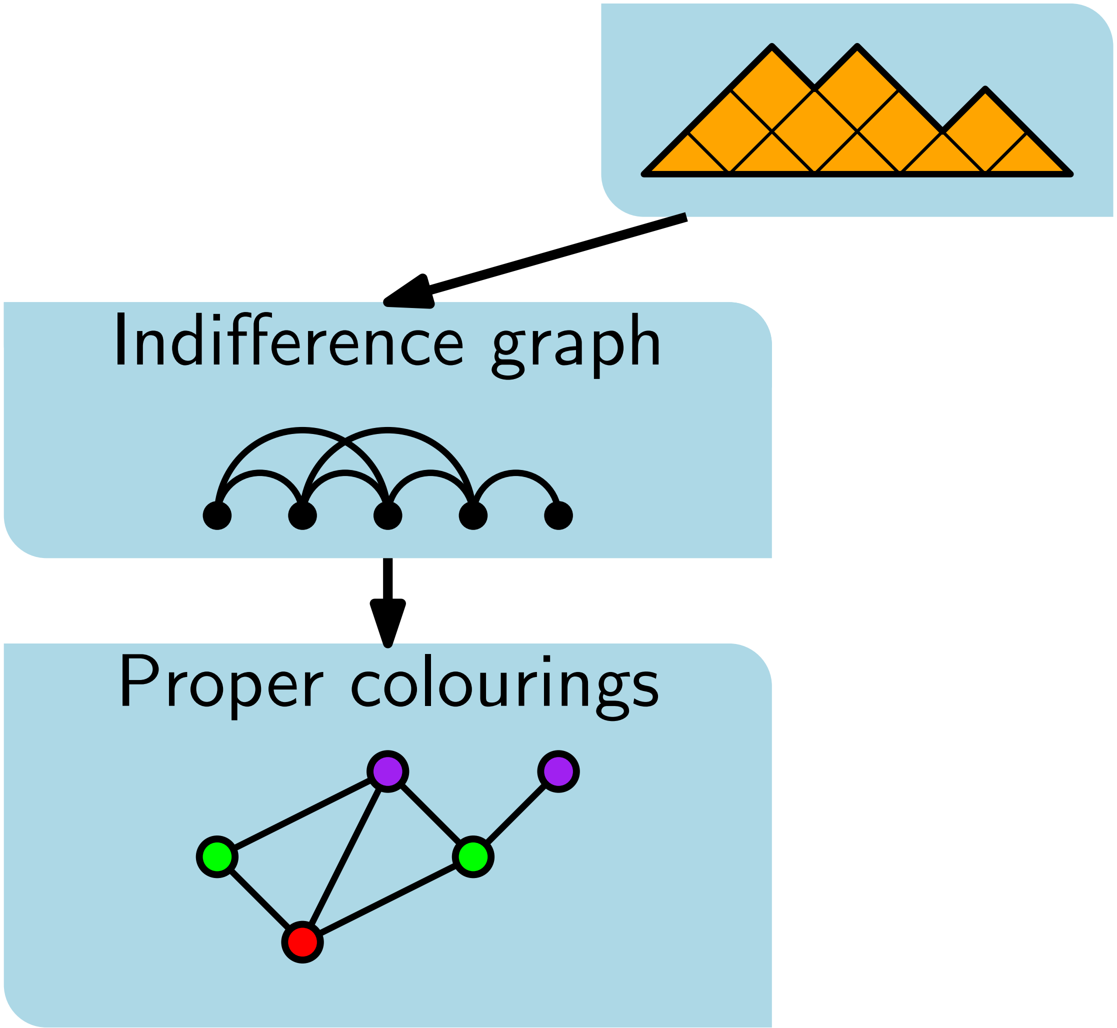

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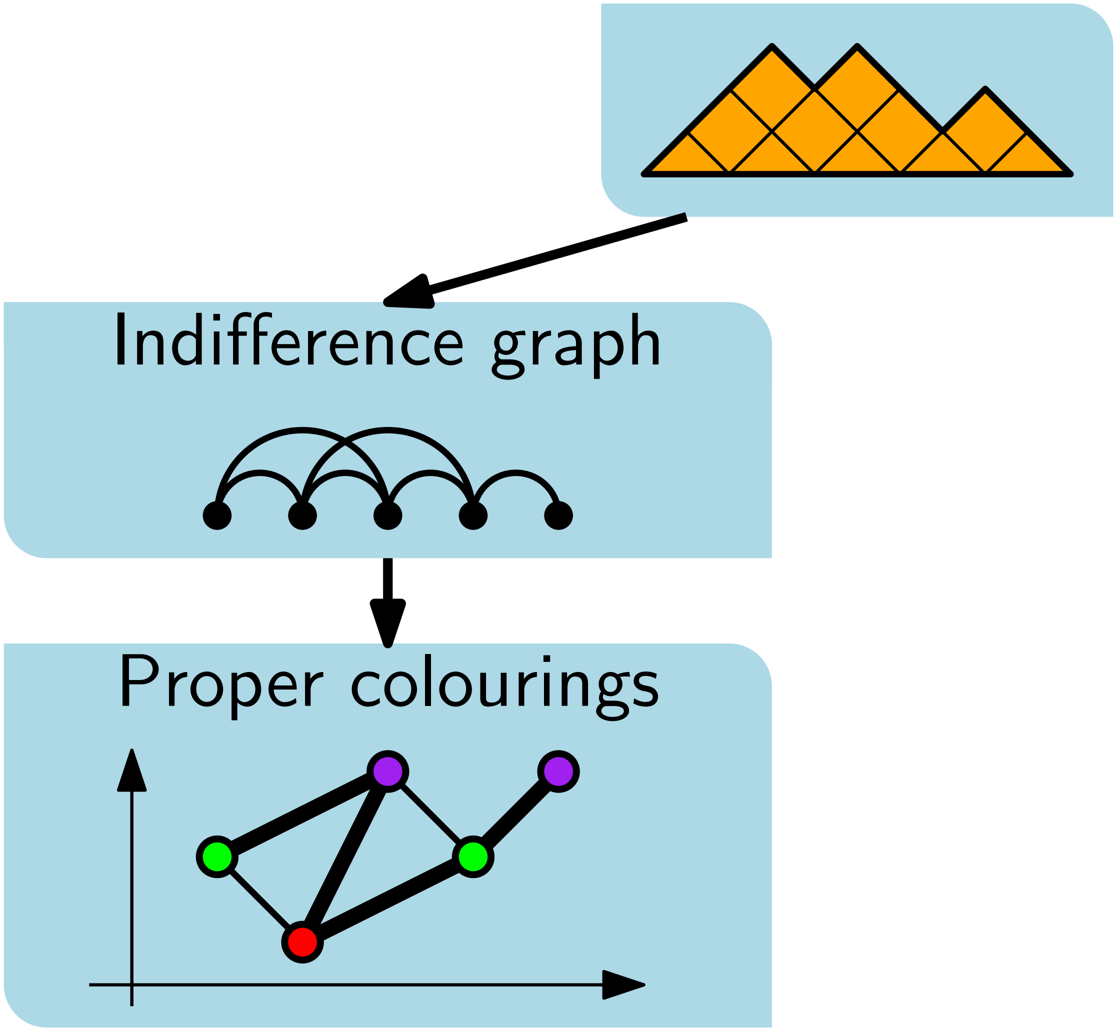

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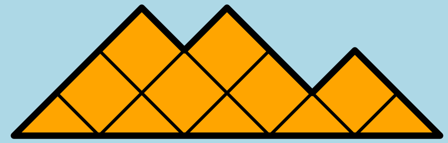

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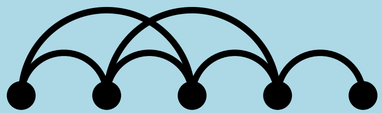
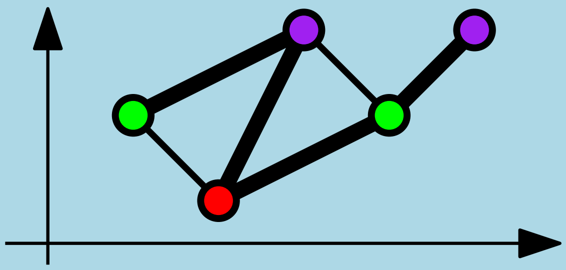


Proper colourings



$$\cdots + q^4 x_1^1 x_2^2 x_3^2 + \cdots$$

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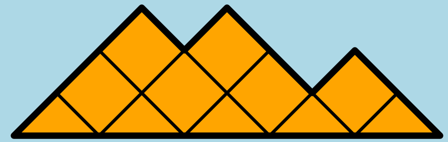

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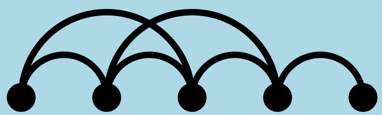

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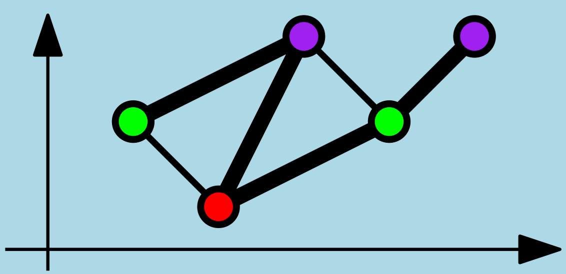

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- Horizontal edge: monochromatic, not allowed
- # vertices at level i: multiplicity, exponent of x_i
- Edge slanting up: ascent, power of q
- Edge slanting down: descent, ignored

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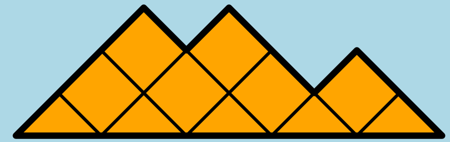

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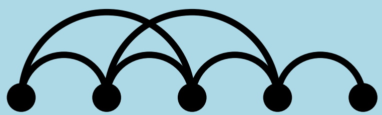

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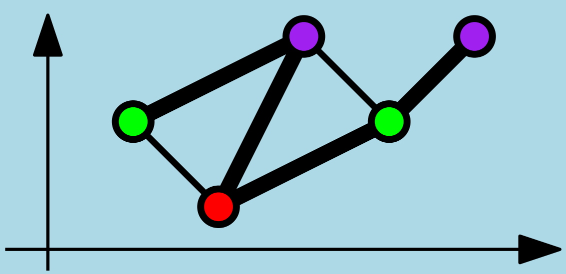

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Birkhoff 1912,
Stanley 1995,
Shareshian–Wachs 2012

The matrix-flag game

Flag $0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 = \mathbb{C}^5$

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Matrix M

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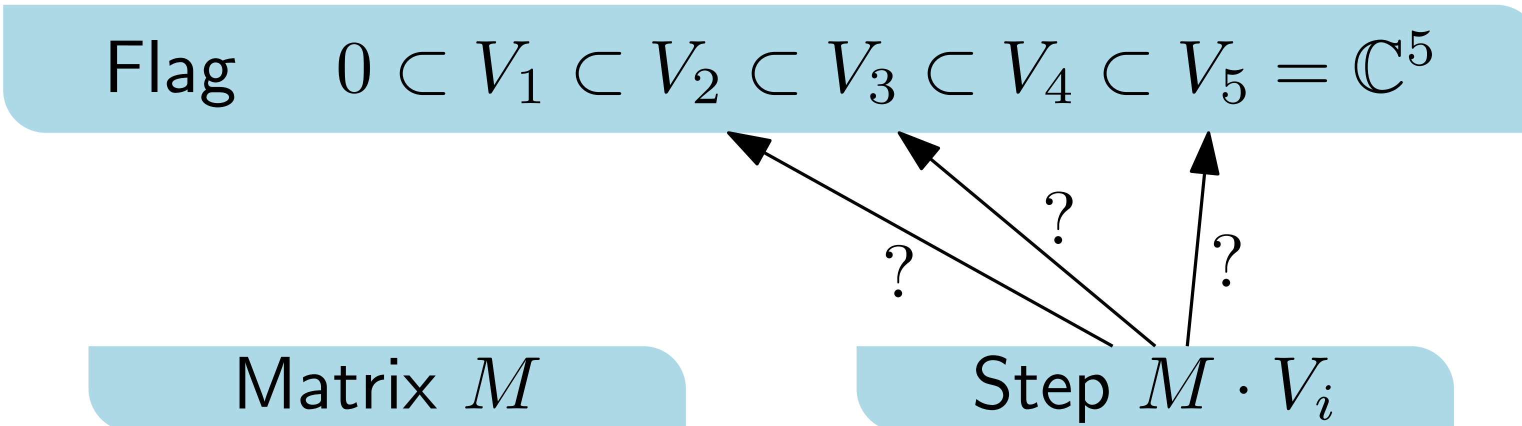
Matrix M

Step $M \cdot V_i$

The matrix-flag game



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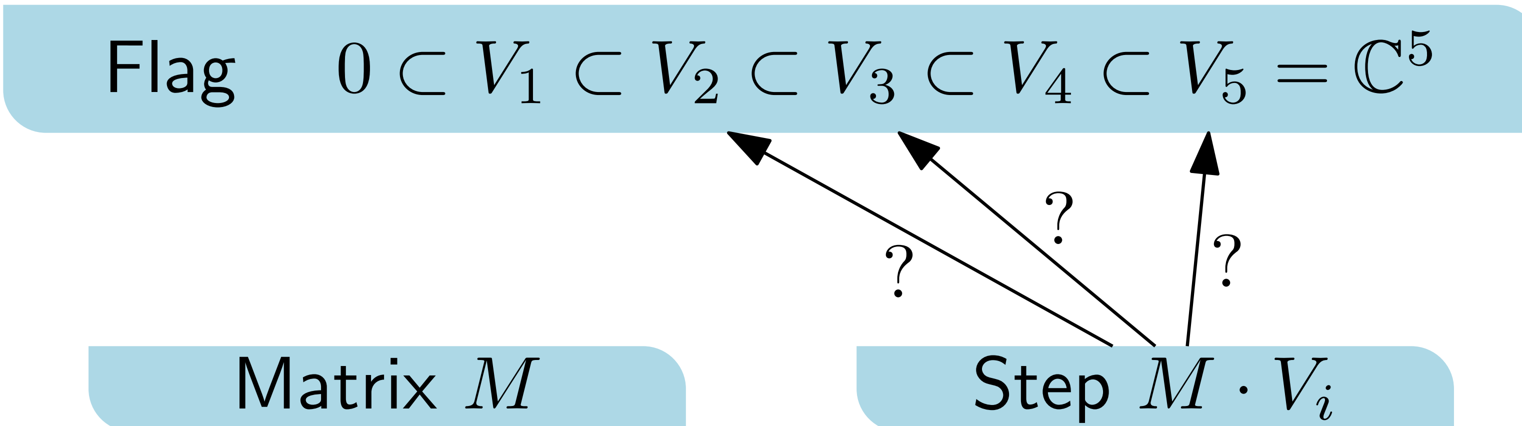
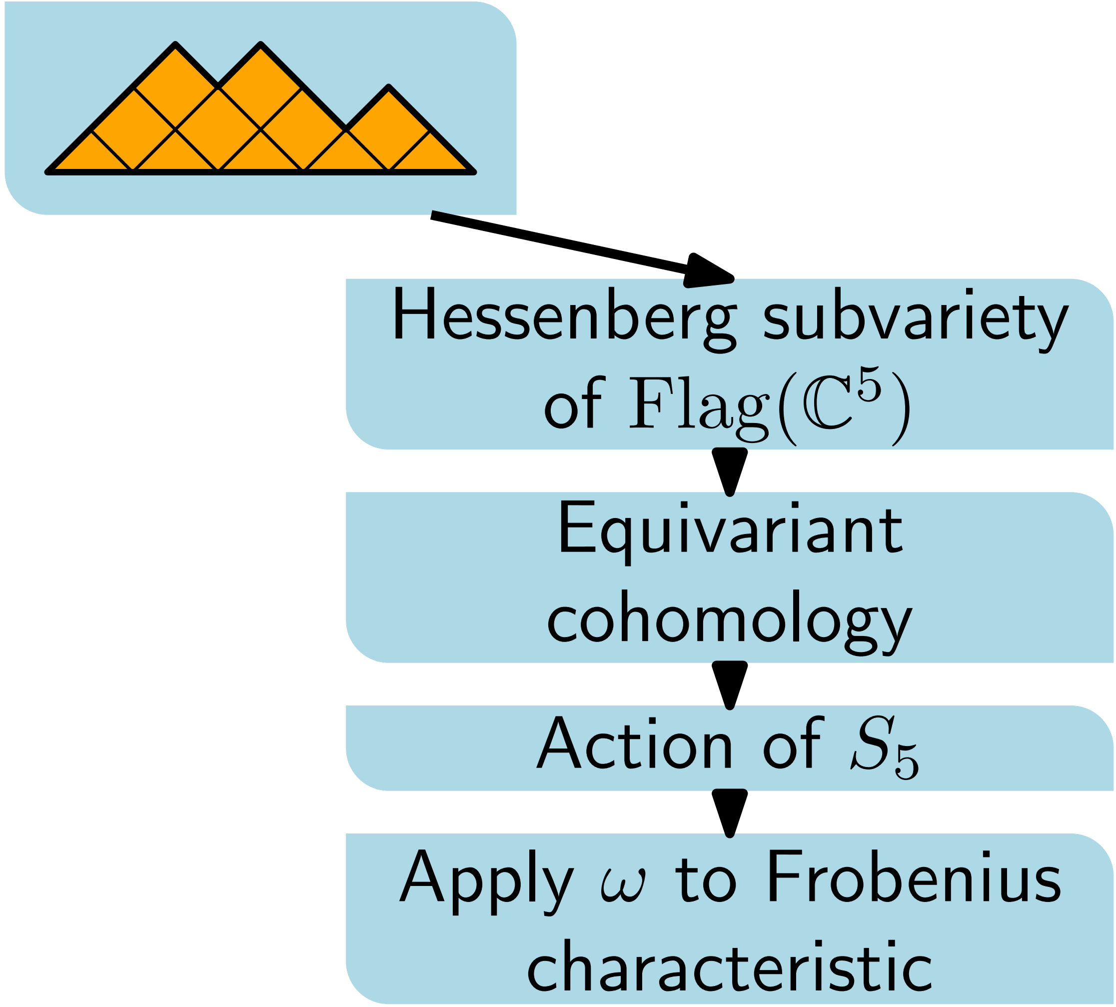


- Invariant under $M \mapsto aM + bI$
- Isomorphic under $M \mapsto BMB^{-1}$
- Jordan blocks and eigenvalues matter
- M has k Jordan blocks
  $\Rightarrow$ it commutes with a k-torus

How it works: flags



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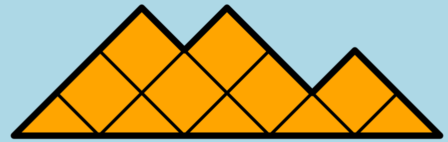

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de Mari–Procesi –Shayman 1988, Tymoczko 2007

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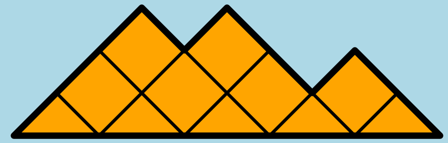

Hessenberg subvariety of $\mathrm{Flag}(\mathbb{C}^5)$

↓

Equivariant cohomology

↓

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Brosnan–Chow 2015: Reciprocity, deform variety

Cohomology

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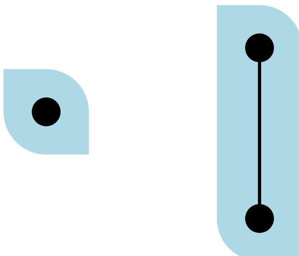

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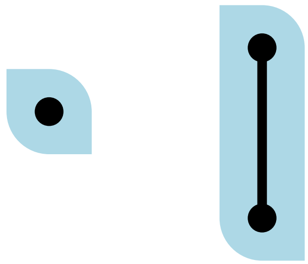

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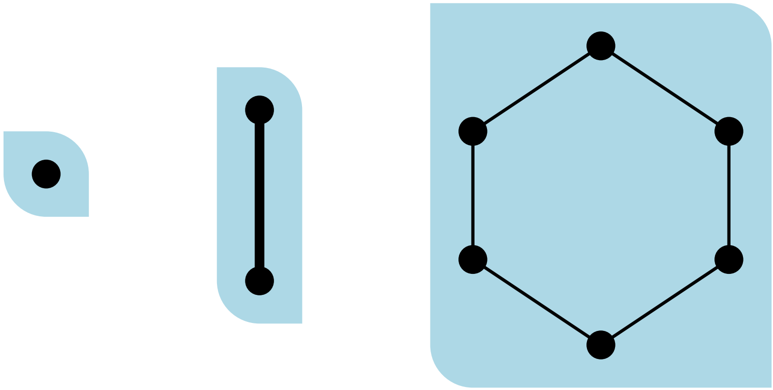

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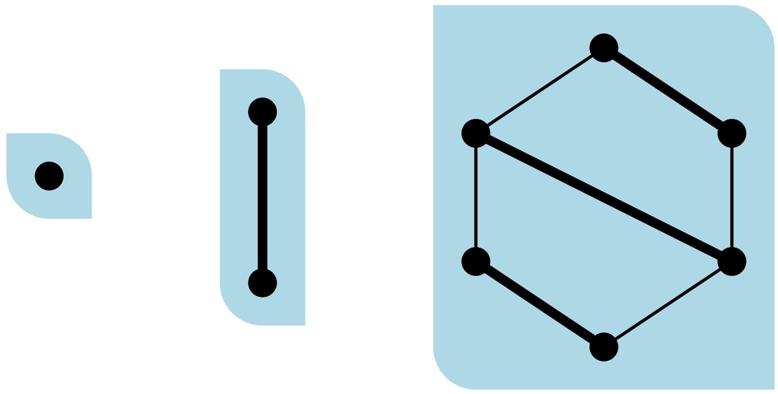

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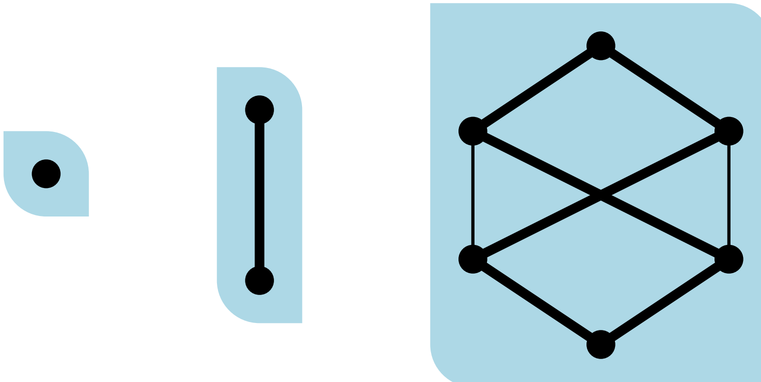

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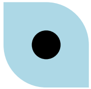
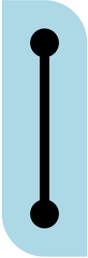
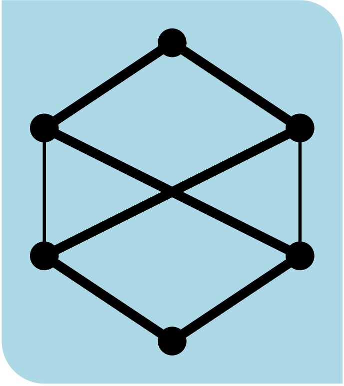
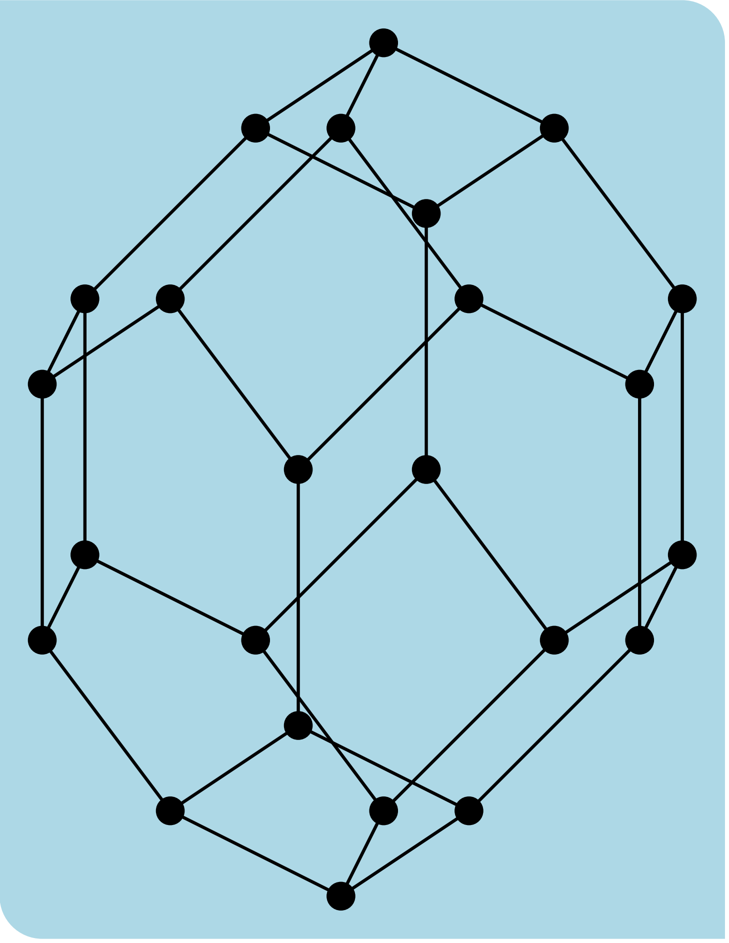

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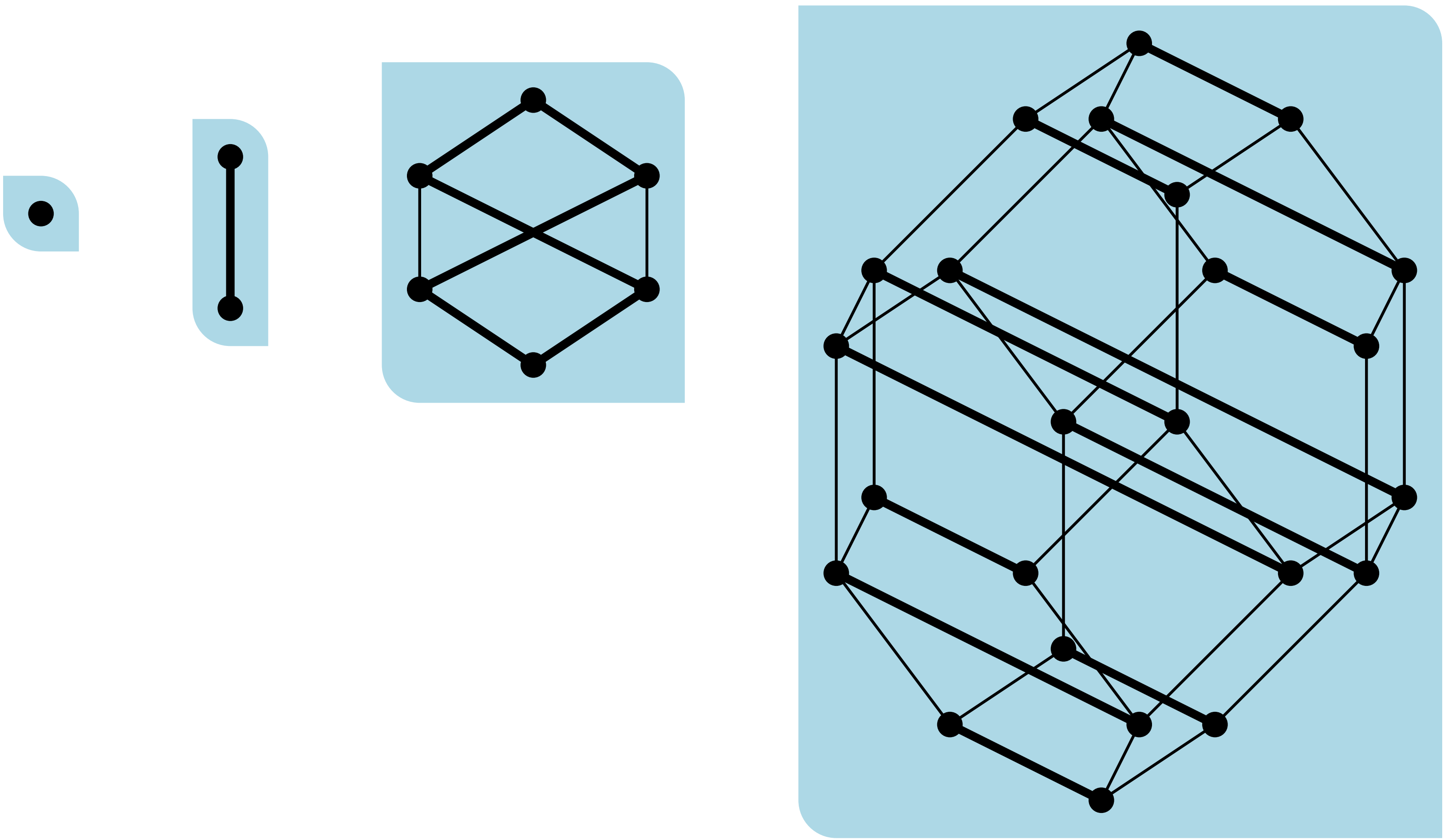

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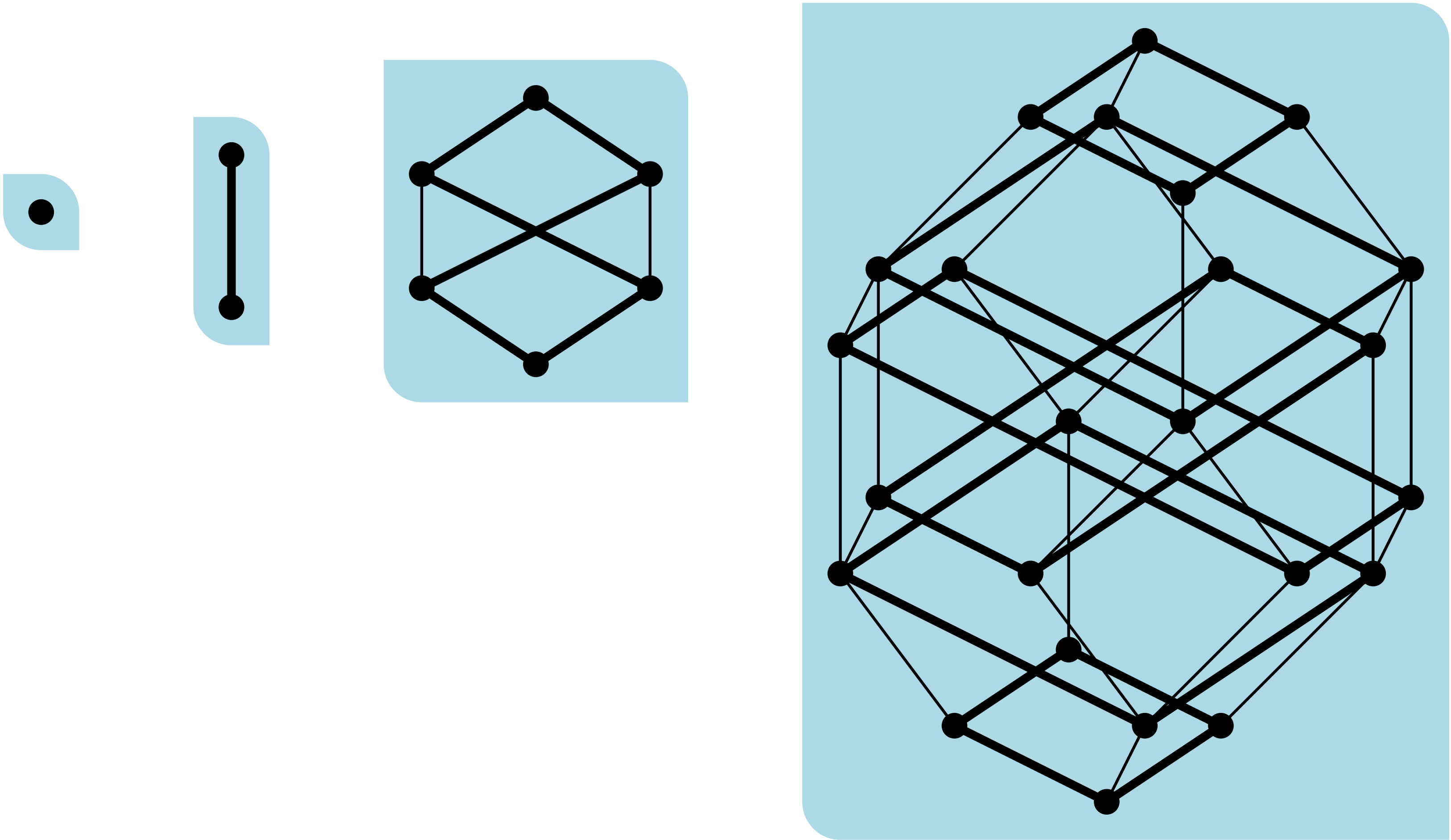

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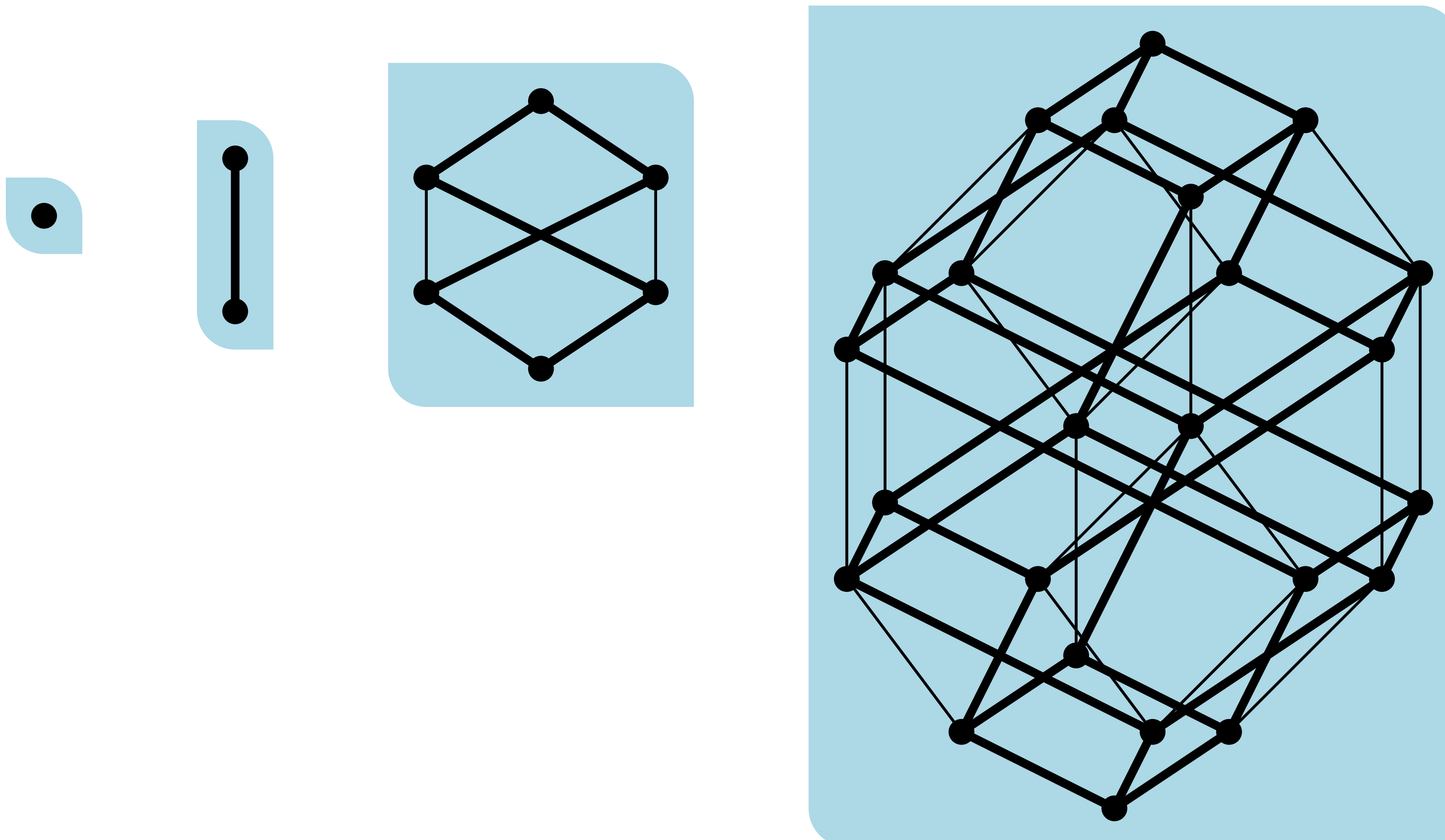

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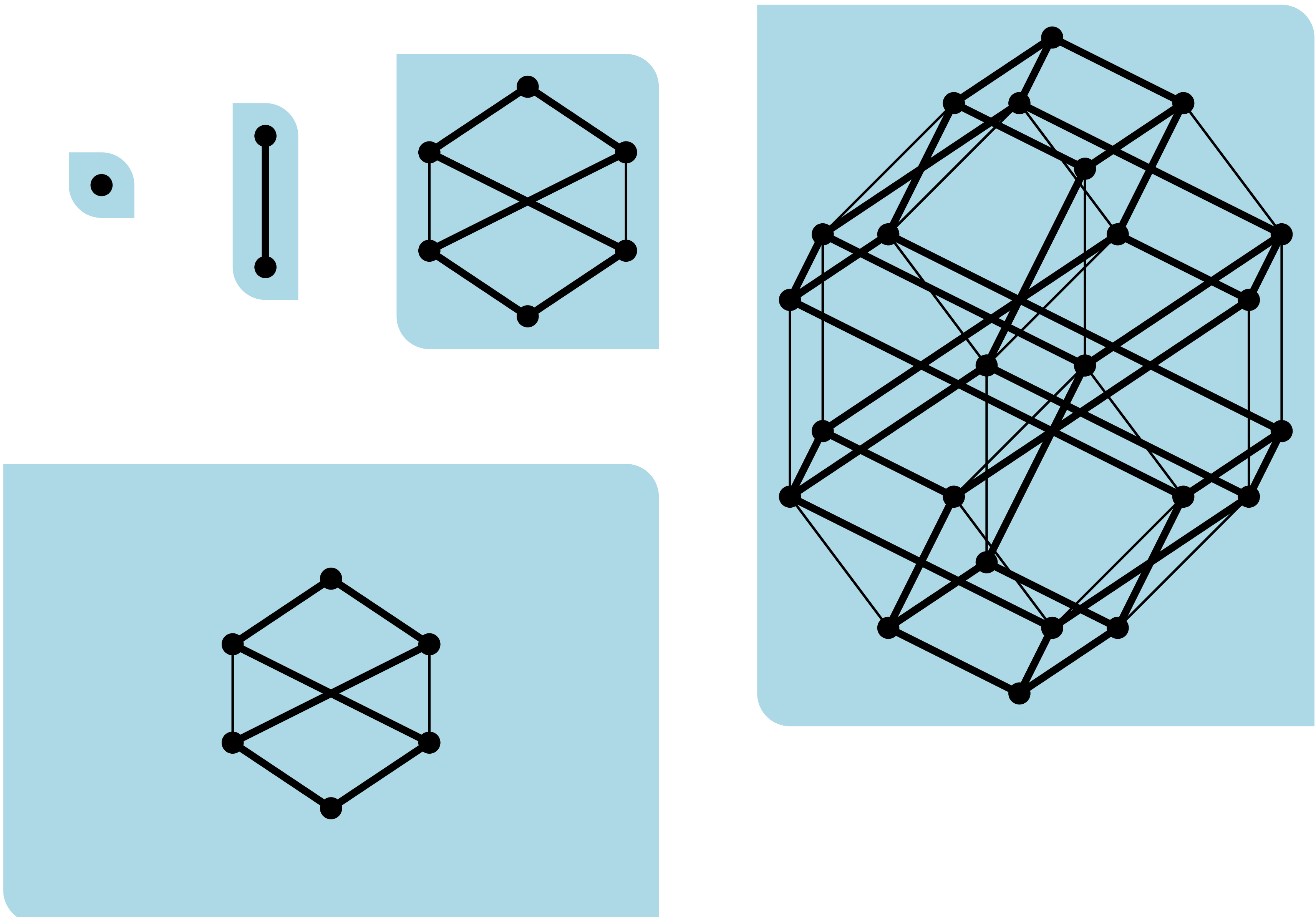

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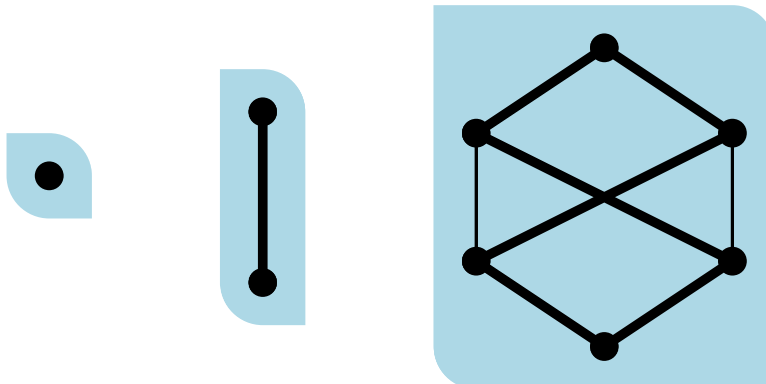

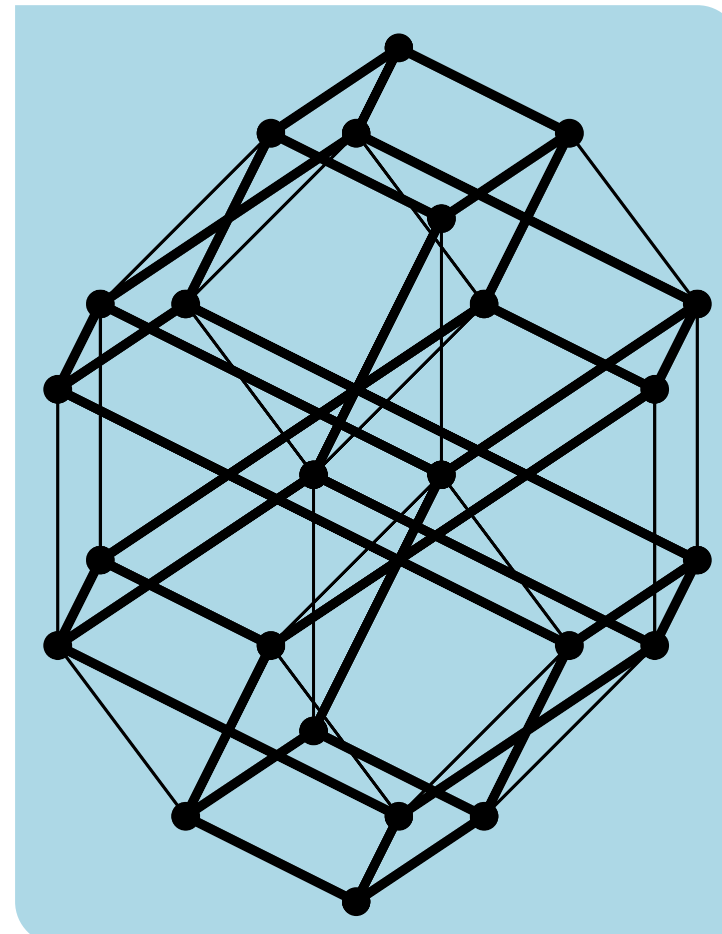

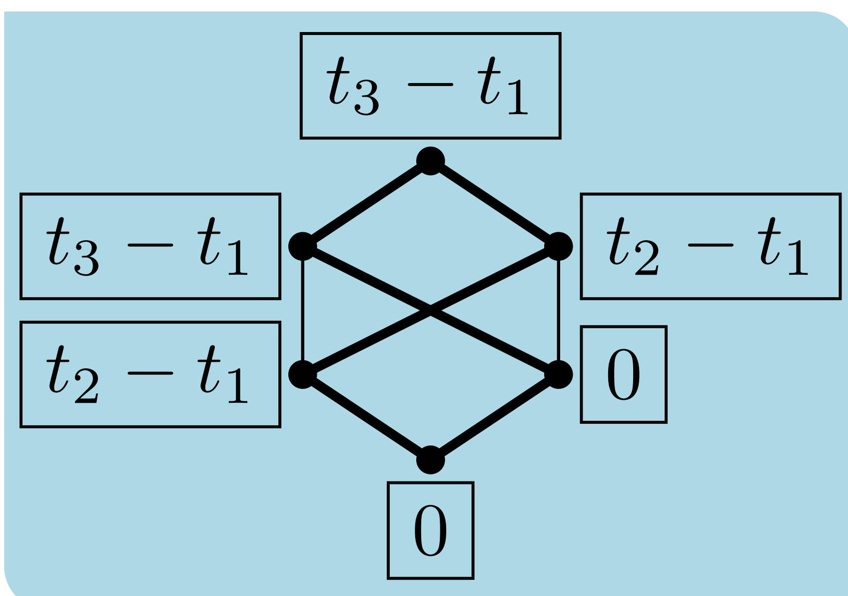

How much work is this?

For every Dyck path
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Trace of a permutation
action on $(\mathbb{Q}[t_1, \ldots, t_n])^{n!}$
(ouf!)

Universal recipe for QSym

(Aguiar–Bergeron–Sottile 2006): If $\mathcal{H}$ is a graded-connected Hopf algebra and ζ is a multiplicative function from $\mathcal{H}$ to the ground ring, then there is a unique map of graded Hopf algebras

$$\Psi_\zeta : \mathcal{H} \to \mathrm{QSym}$$

which sends ζ to ζ_Q. The coefficient of M_α in $\Psi_\zeta(h)$ is

$$(\underbrace{\zeta \otimes \zeta \otimes \cdots \otimes \zeta}_{r \text{ copies}}) \circ (\pi_{\alpha_1} \otimes \pi_{\alpha_2} \otimes \cdots \otimes \pi_{\alpha_r}) \circ \Delta_r(h),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a list of r positive integers, Δ_r is the r-fold comultiplication map of $\mathcal{H}$, and π_n is the projection onto the homogeneous part of degree n.

How much work does this save?

For every Dyck path
($\approx 4^n$)

For every integer partition
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→ For every Dyck path
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For every integer partition
($\approx c^{\sqrt{n}}$)

→ For the partition (n)
($= 1$)

Count proper colourings
($\leq n!$)

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How much work does this save?

For every Dyck path ($\approx 4^n$) → For every Dyck path ($\approx 4^n$)

For every integer partition ($\approx c^{\sqrt{n}}$) → For the partition (n) ($= 1$)

Count proper colourings ($\leq n!$) → Count proper 1-colourings (≤ 1)

Trace of a permutation action on $(\mathbb{Q}[t_1, \ldots, t_n])^{n!}$ (ouf!)

How much work does this save?

For every Dyck path ($\approx 4^n$)	→	For every Dyck path ($\approx 4^n$)
For every integer partition ($\approx c^{\sqrt{n}}$)	→	For the partition (n) ($= 1$)
Count proper colourings ($\leq n!$)	→	Count proper 1-colourings (≤ 1)
Trace of a permutation action on $(\mathbb{Q}[t_1, \ldots, t_n])^{n!}$ (ouf!)	→	Dimension of the alternating sub-rep (hmm)

My theorems

- There is a Hopf algebra of Dyck paths

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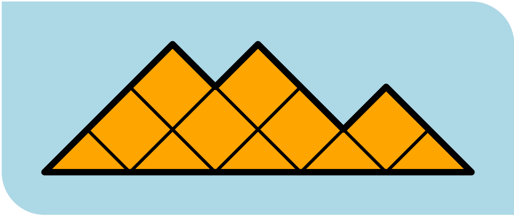
- Both constructions have the same character ζ:

$$\zeta(\mathrm{path}) = \begin{cases} 1 & \mathrm{path}\ \text{has no boxes} \\ 0 & \mathrm{path}\ \text{has any boxes} \end{cases}$$

The Hopf algebra



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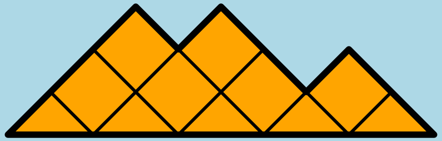

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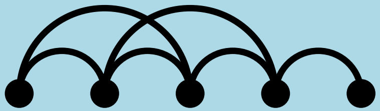

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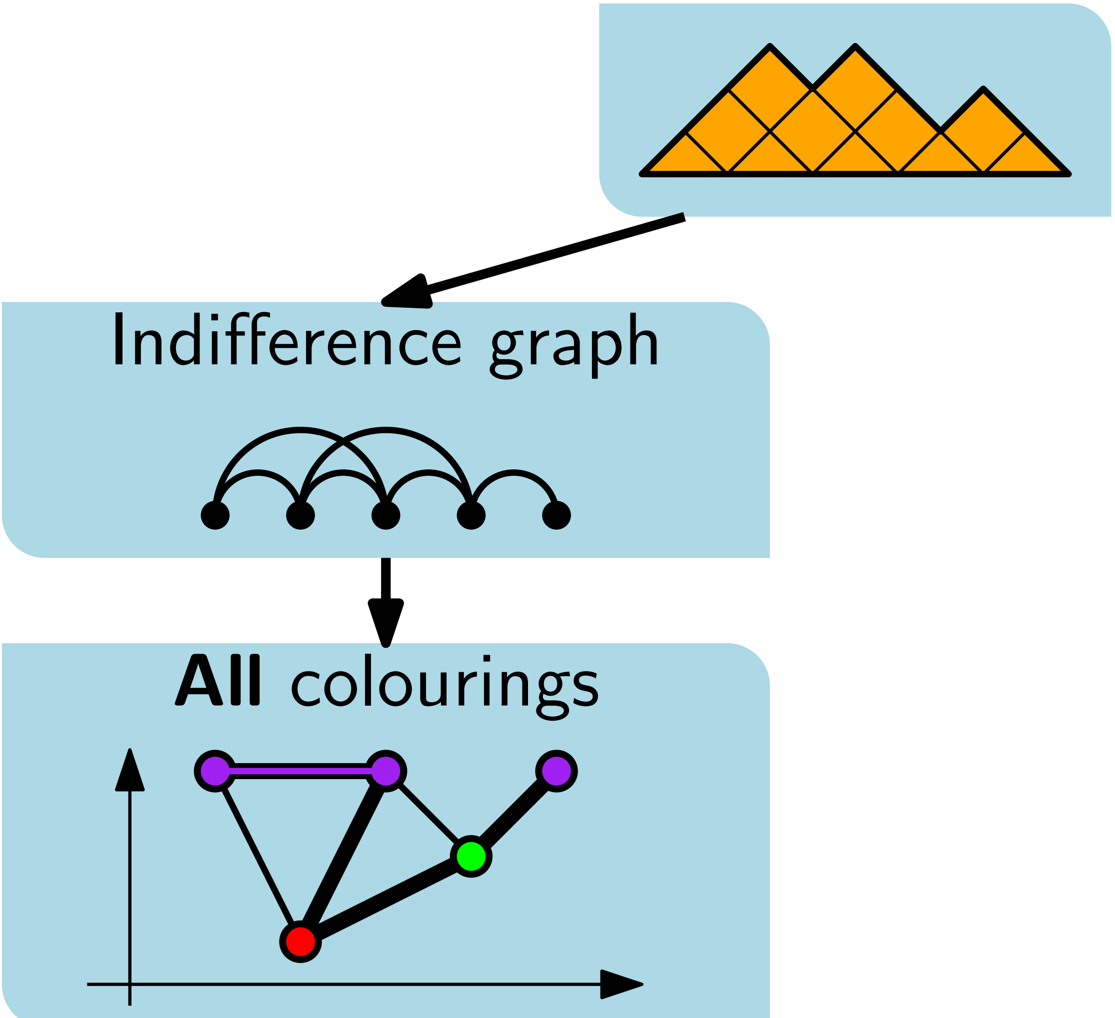
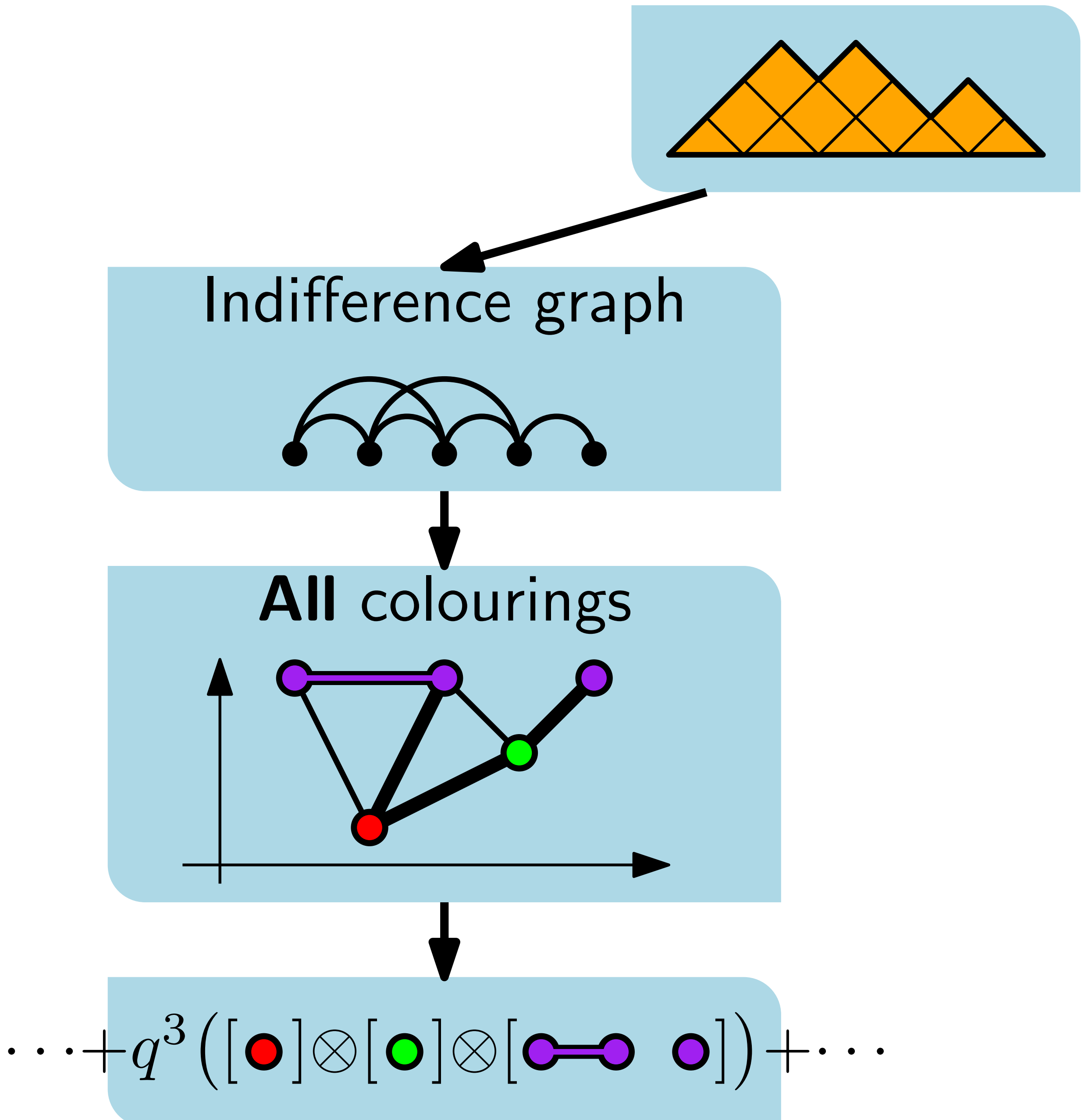


The Hopf algebra



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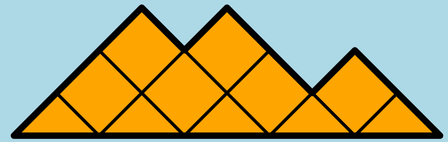
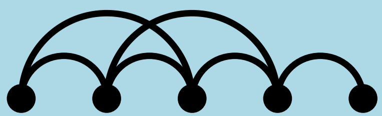
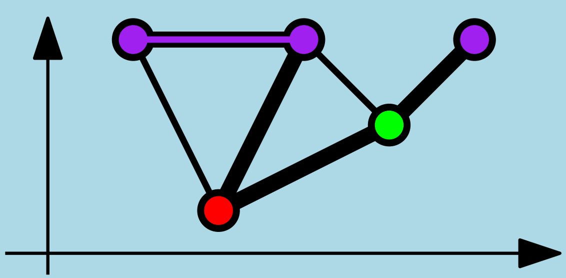
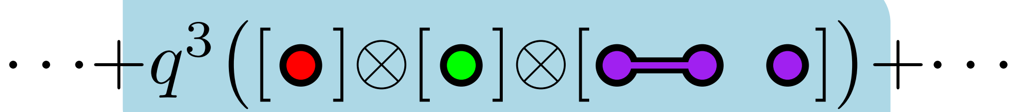


Indifference graph



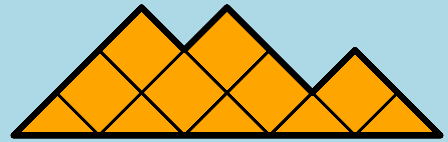
All colourings





- Horizontal edge: monochromatic, **survives**
- Vertices at level i: in ith graph
- Edge slanting up: ascent, power of q
- Edge slanting down: descent, deleted

The Hopf algebra



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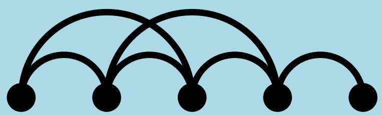

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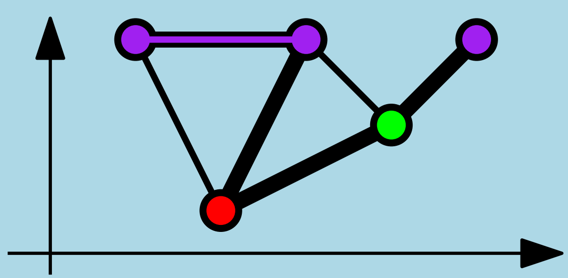

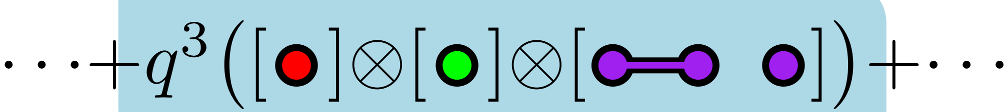

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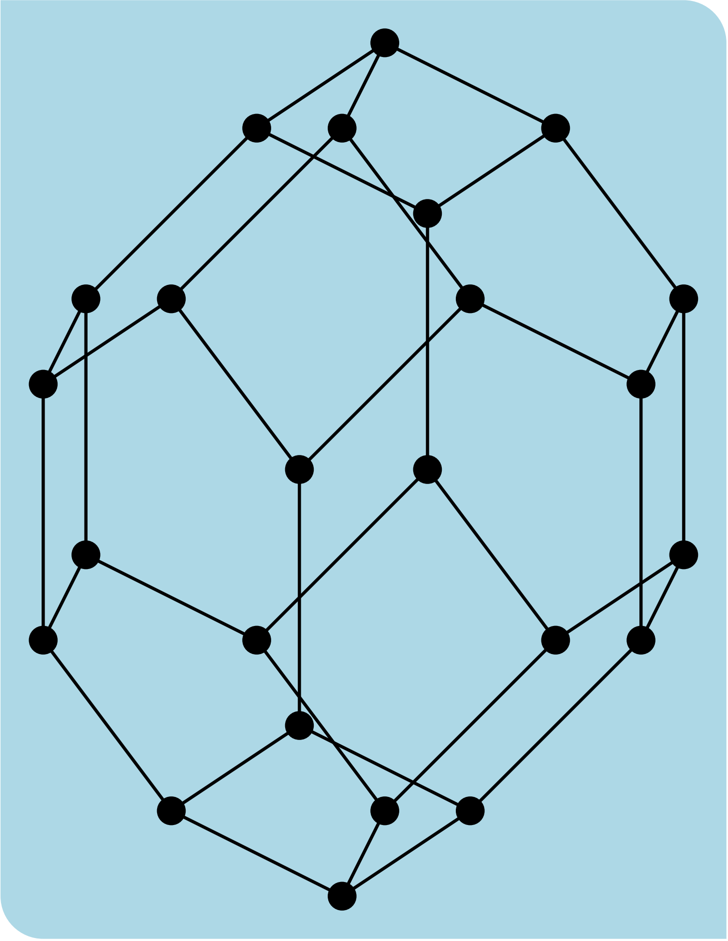
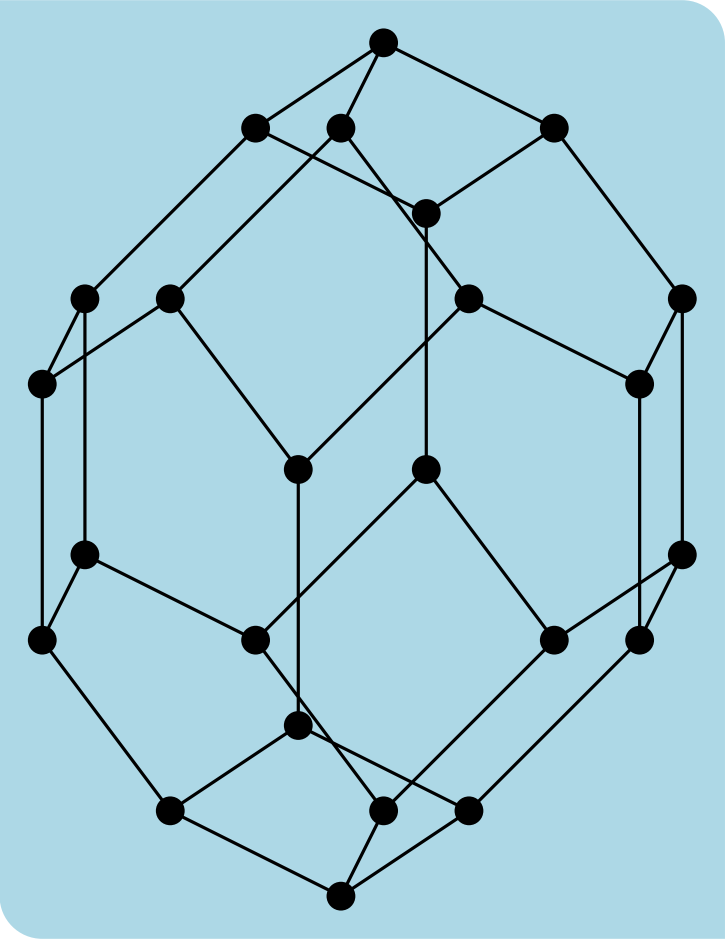
Schmitt 1994,
Athanasiadis 2015

Decompositions





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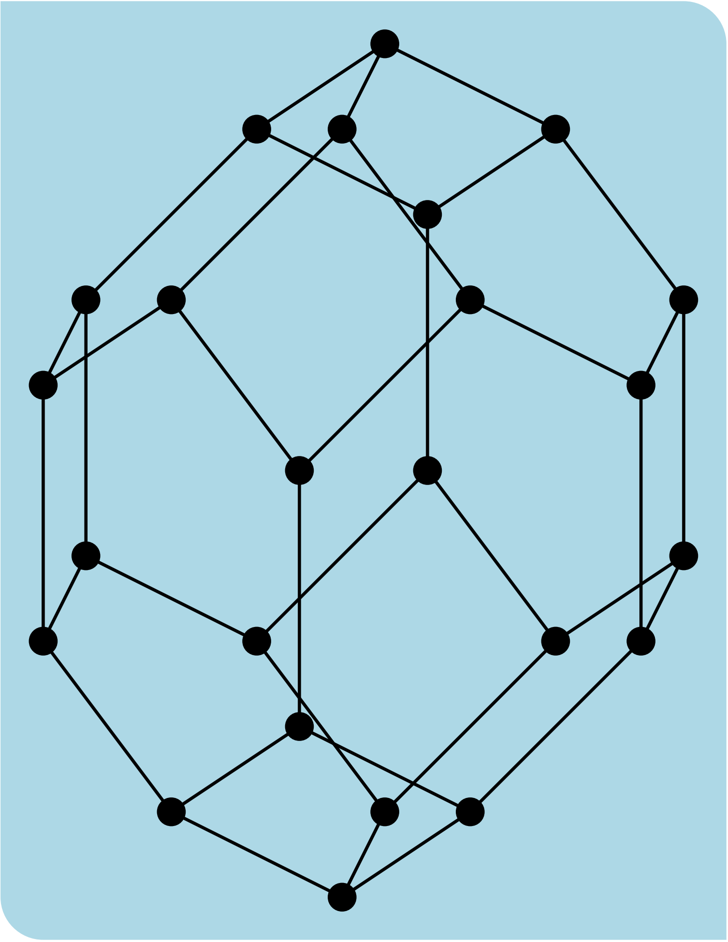
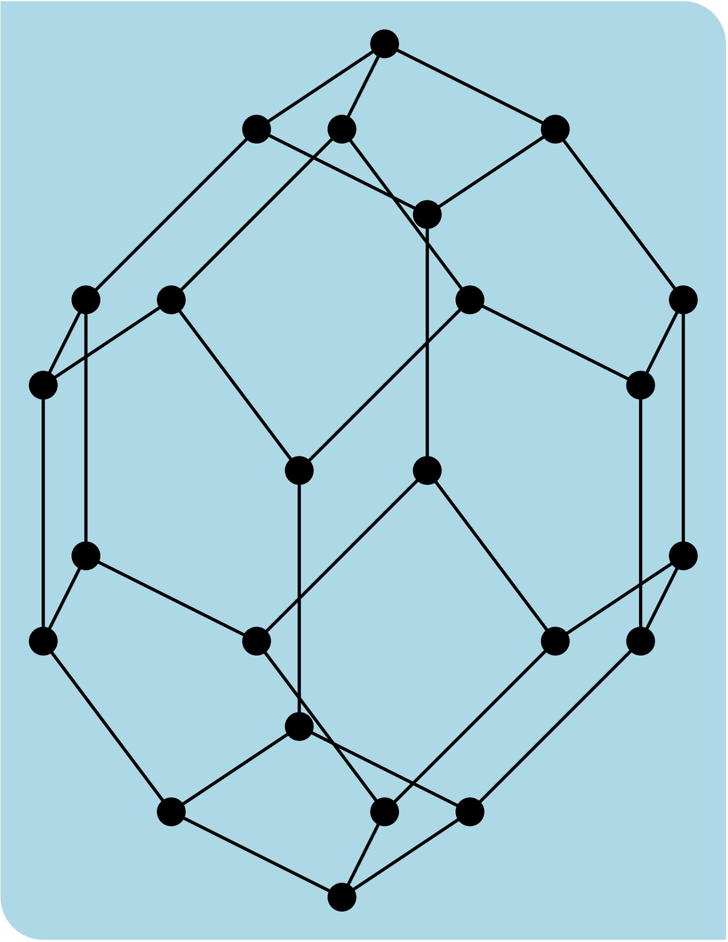
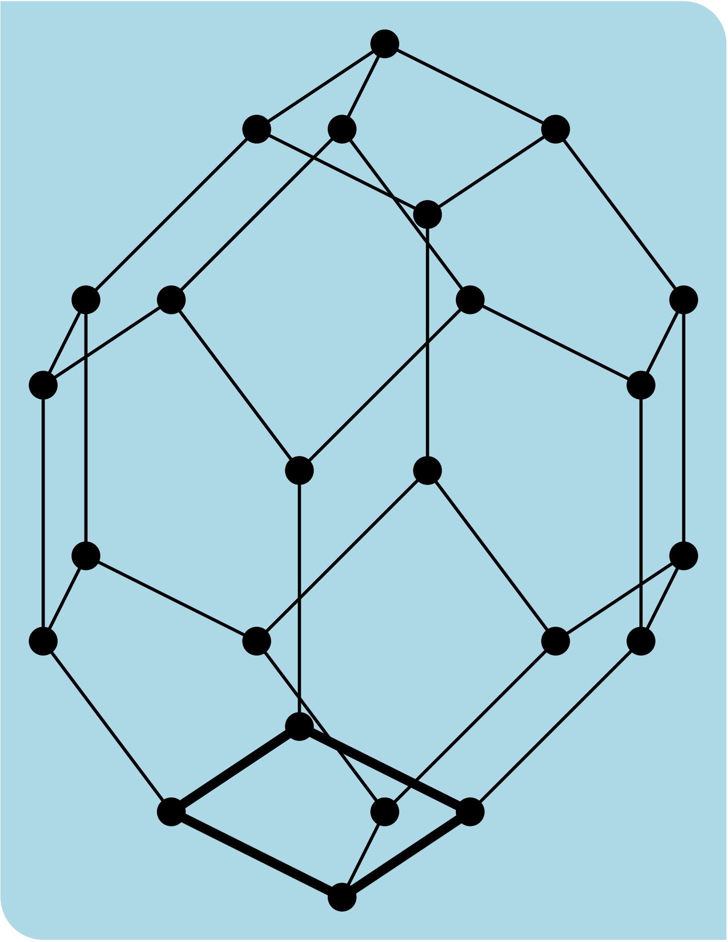
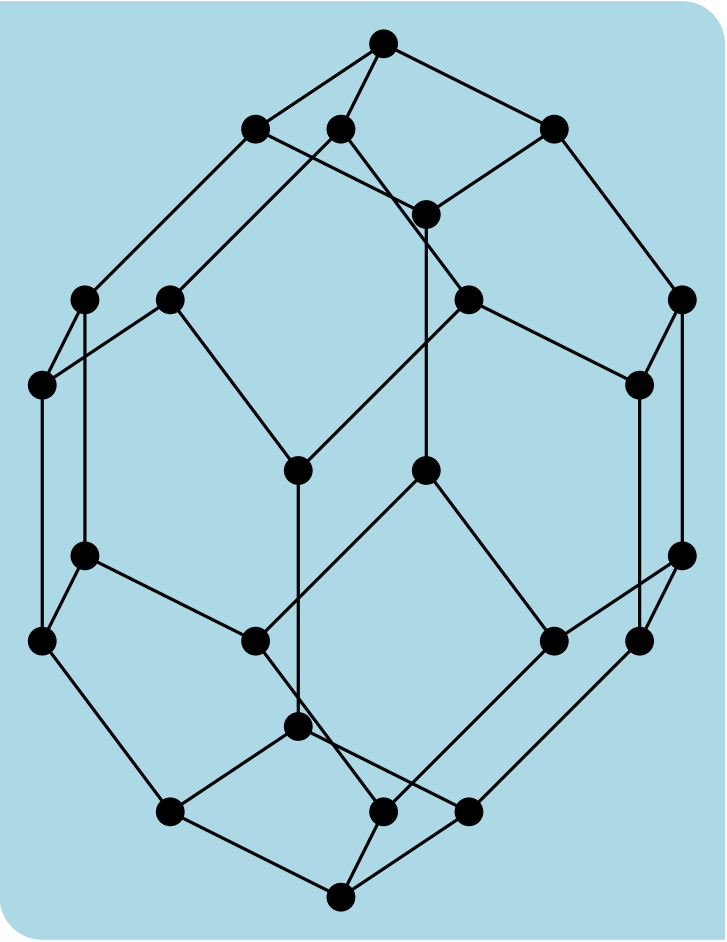




Orbits under $S_2 \times S_2$ from the left and from the right

Decompositions





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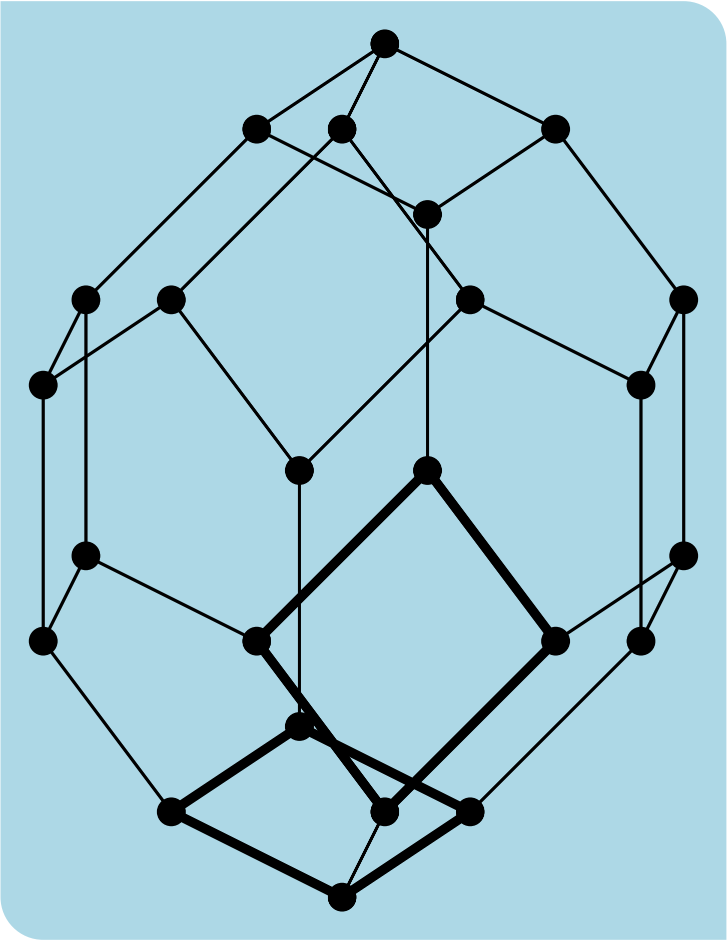

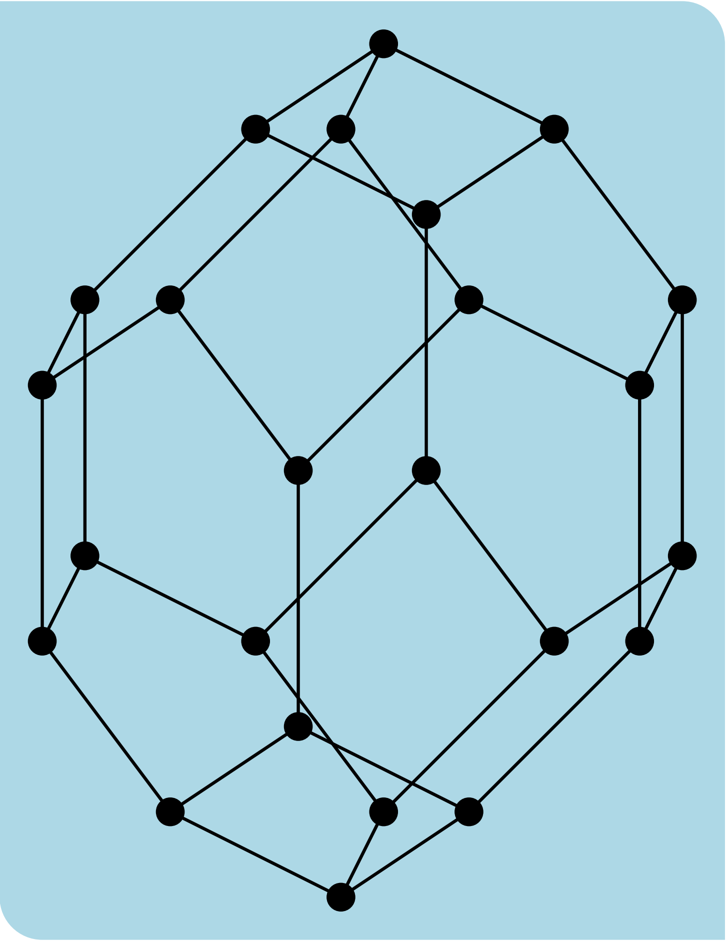

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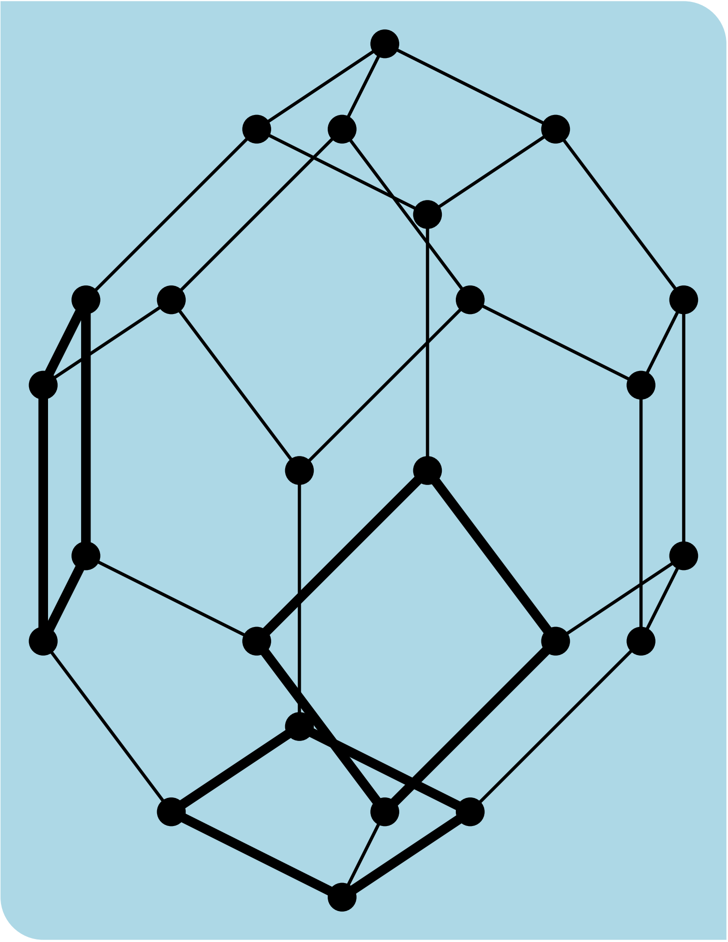

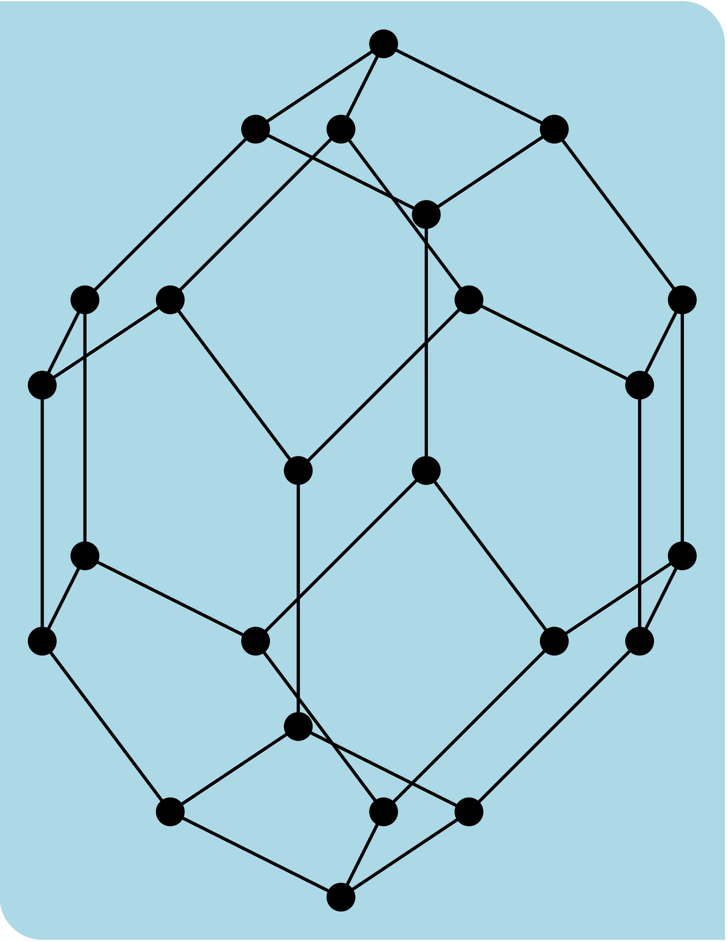

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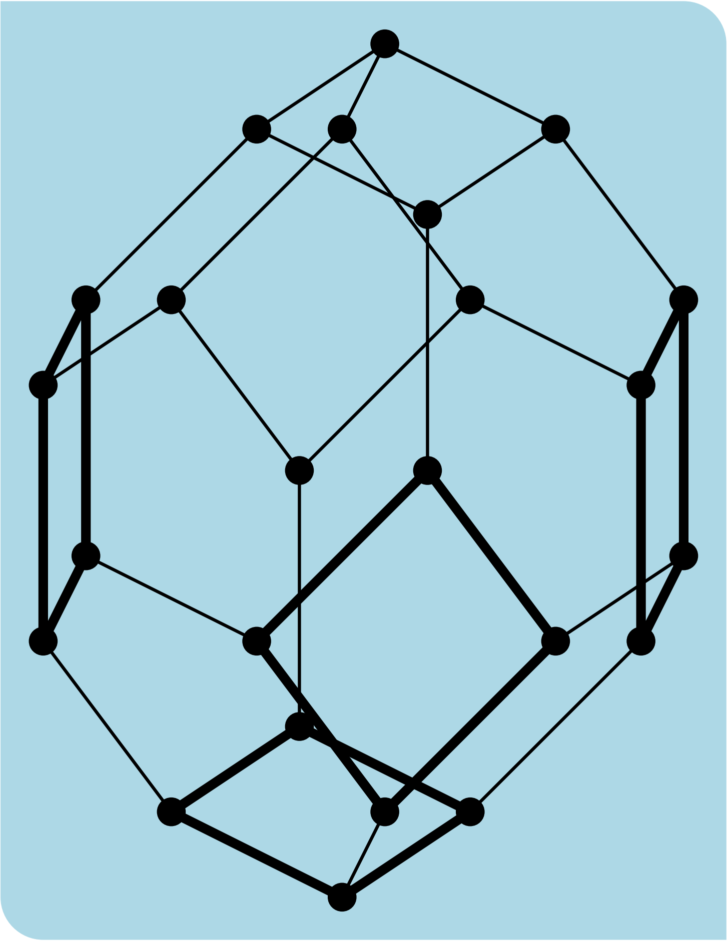

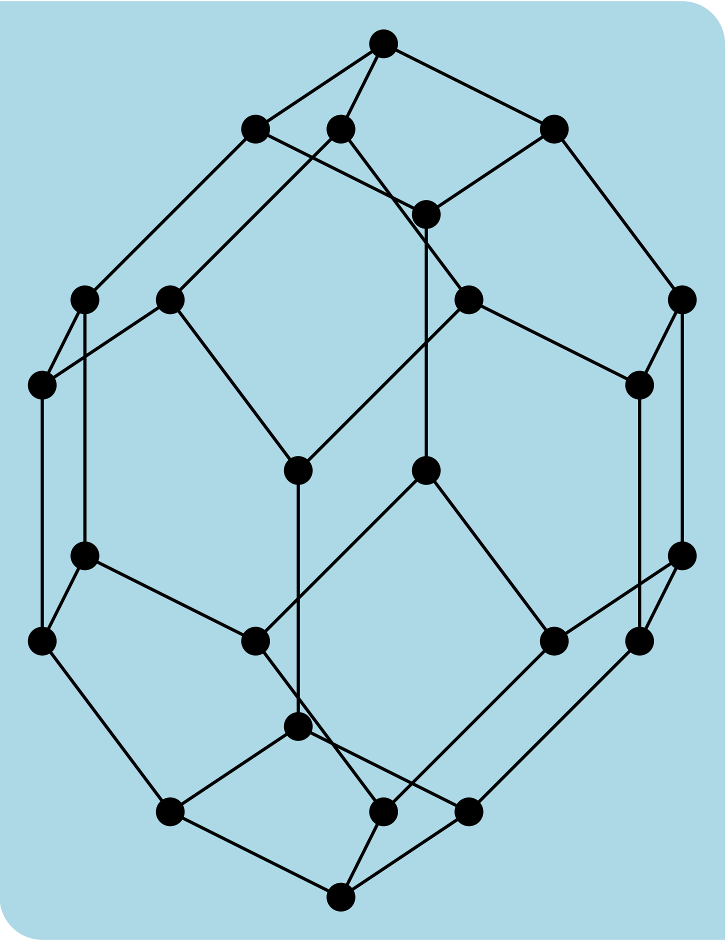

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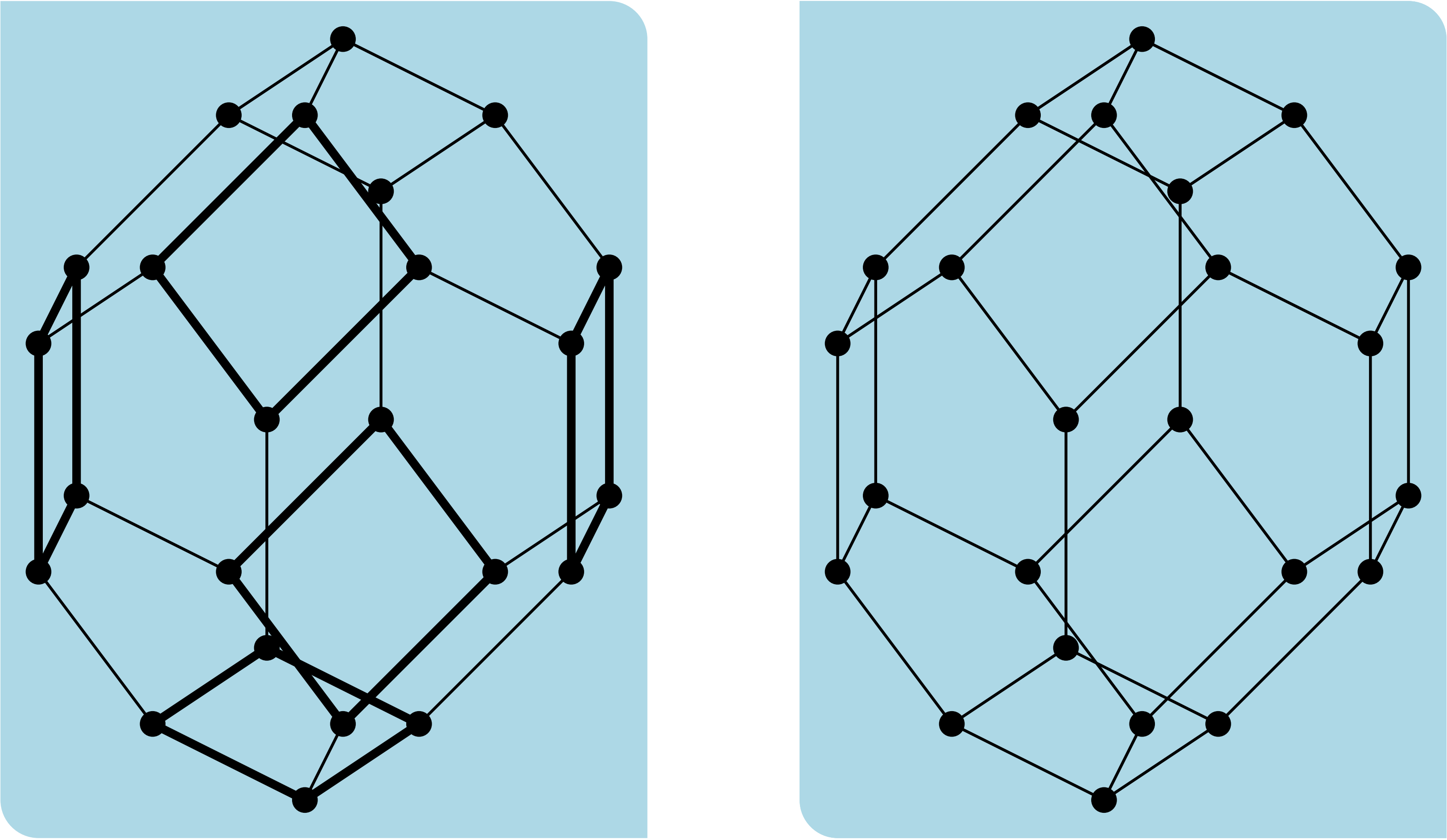

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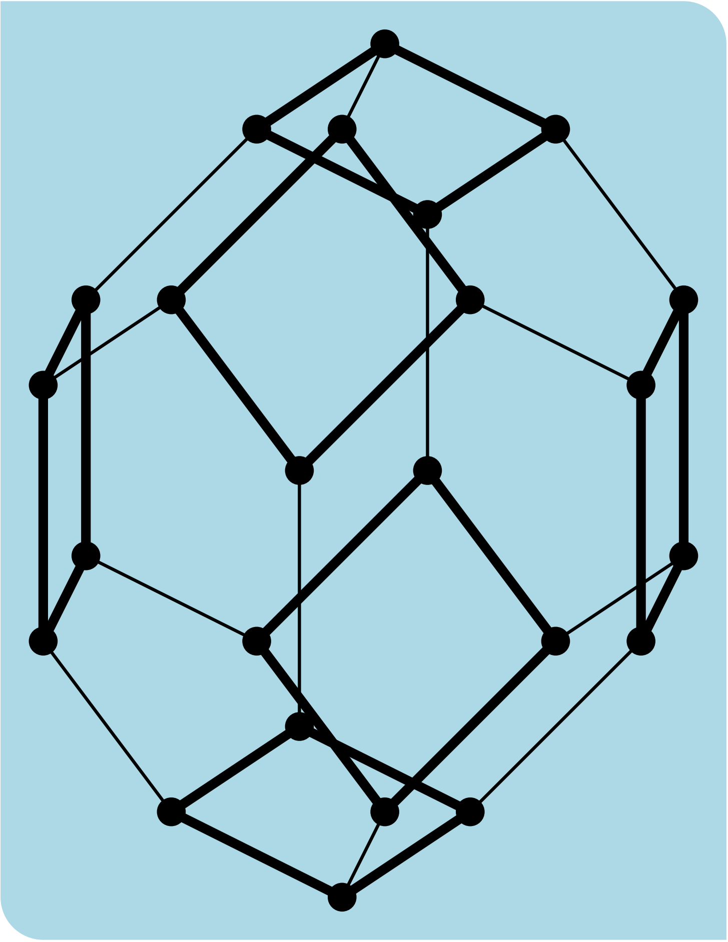

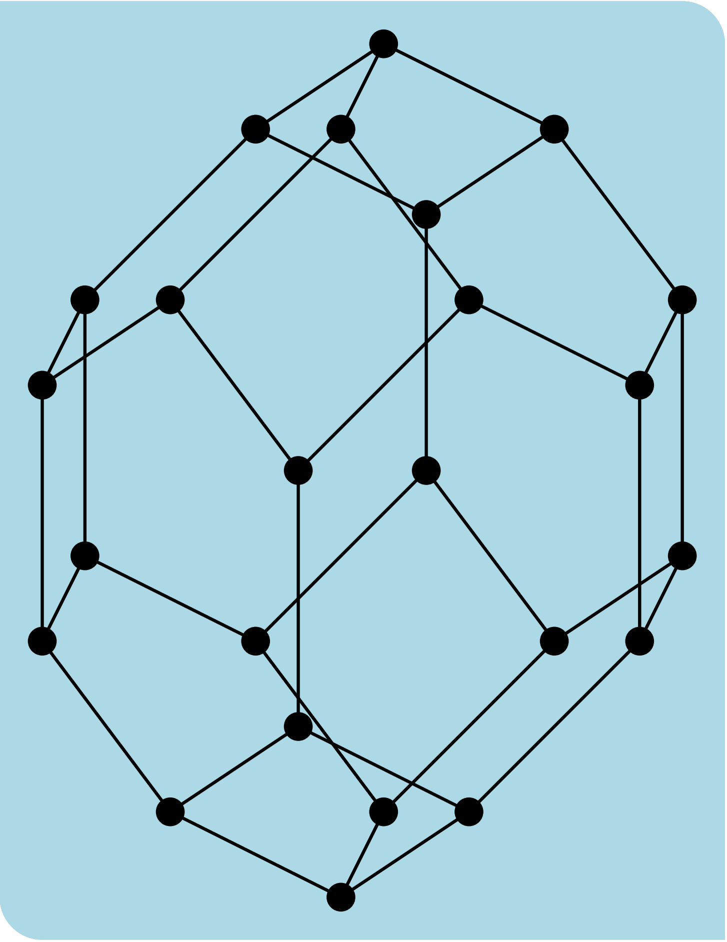

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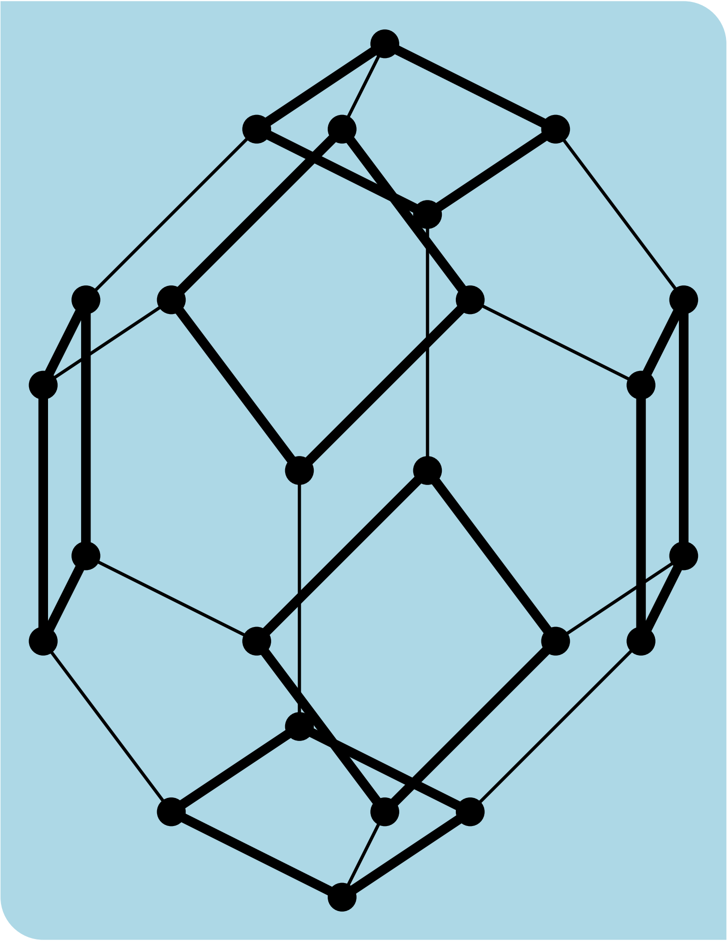

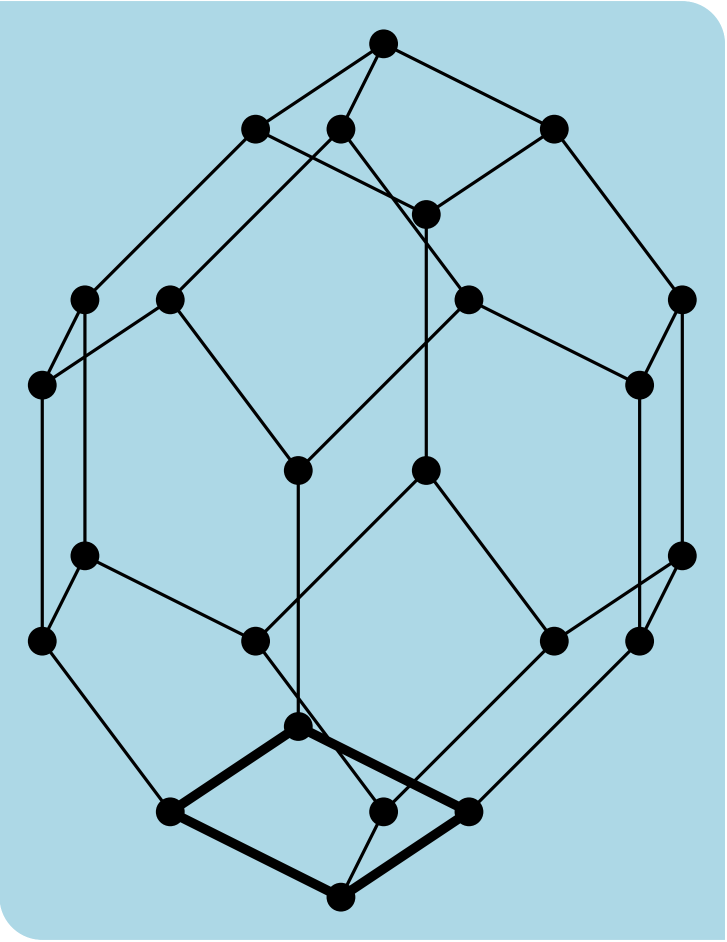

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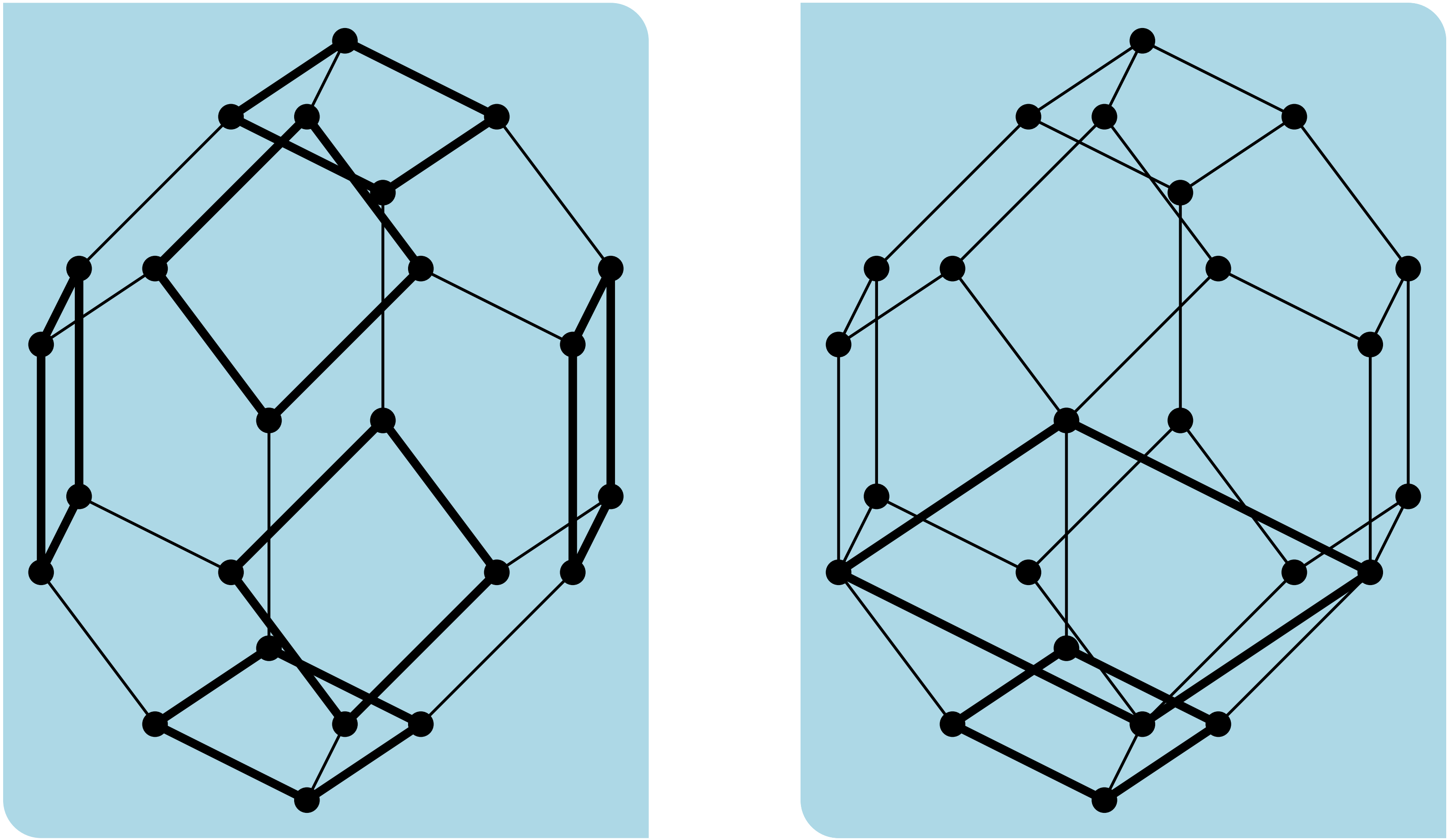

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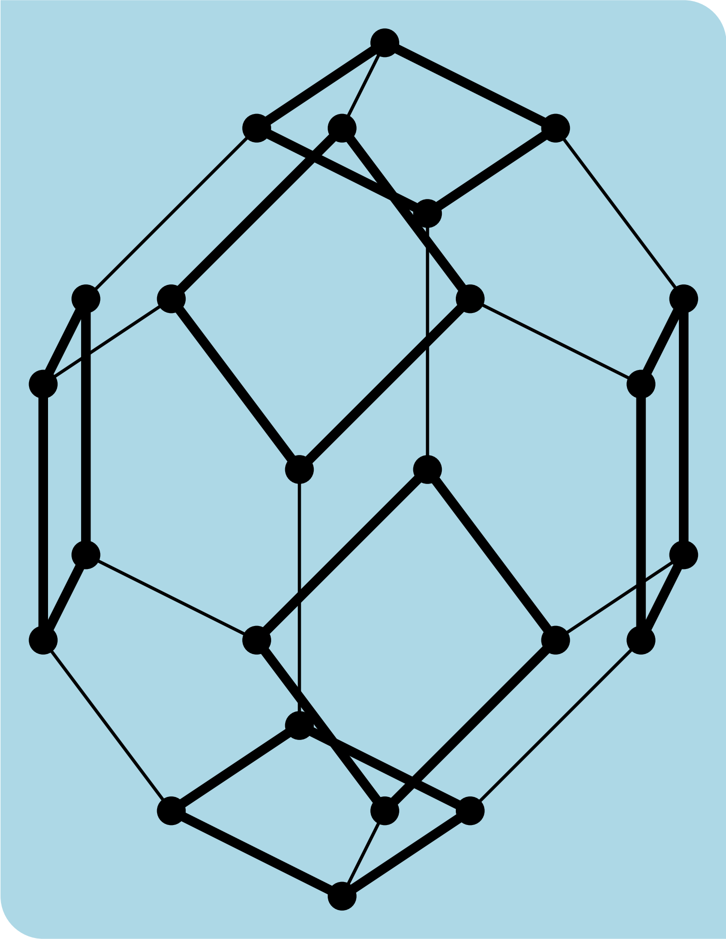

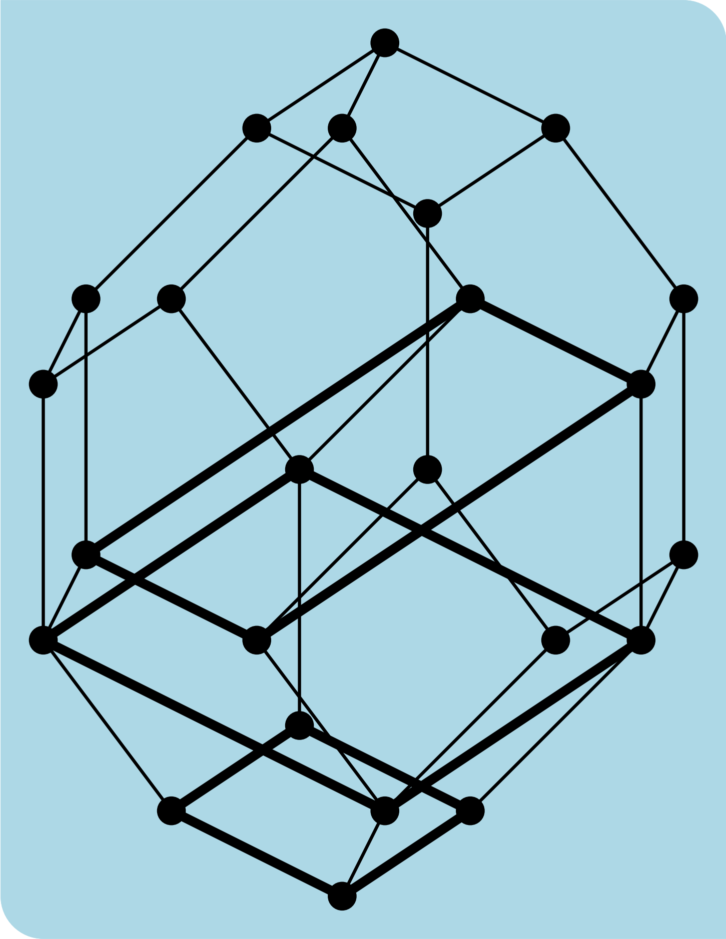

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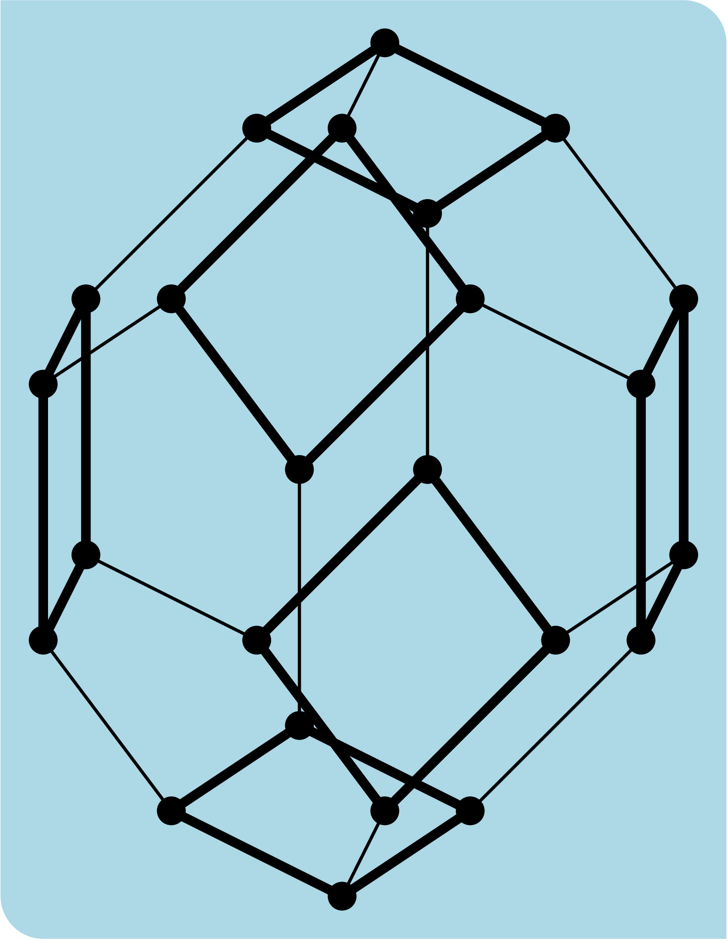

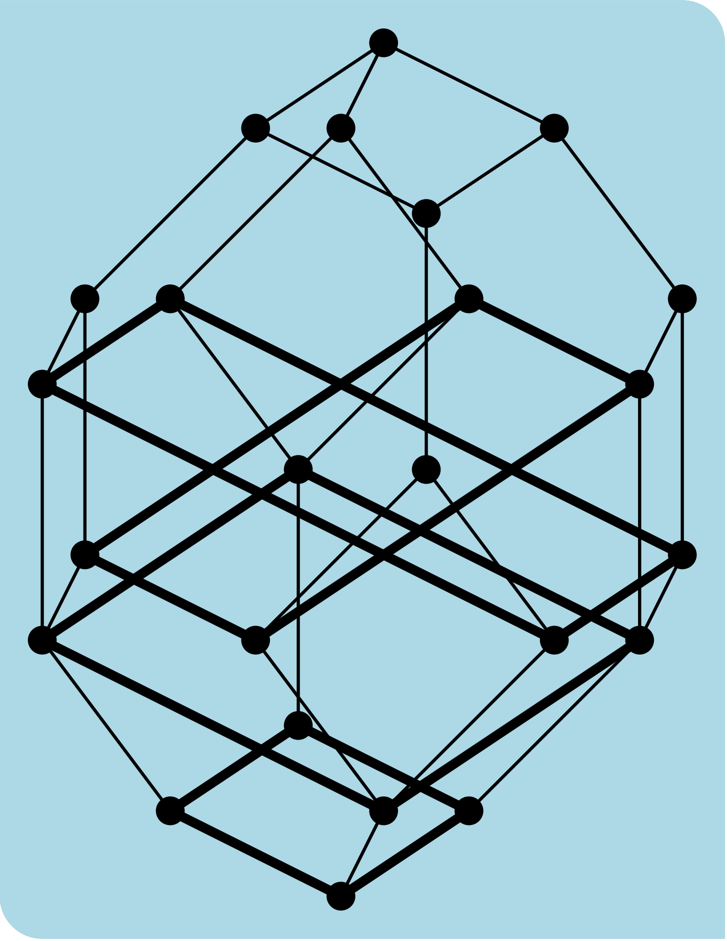

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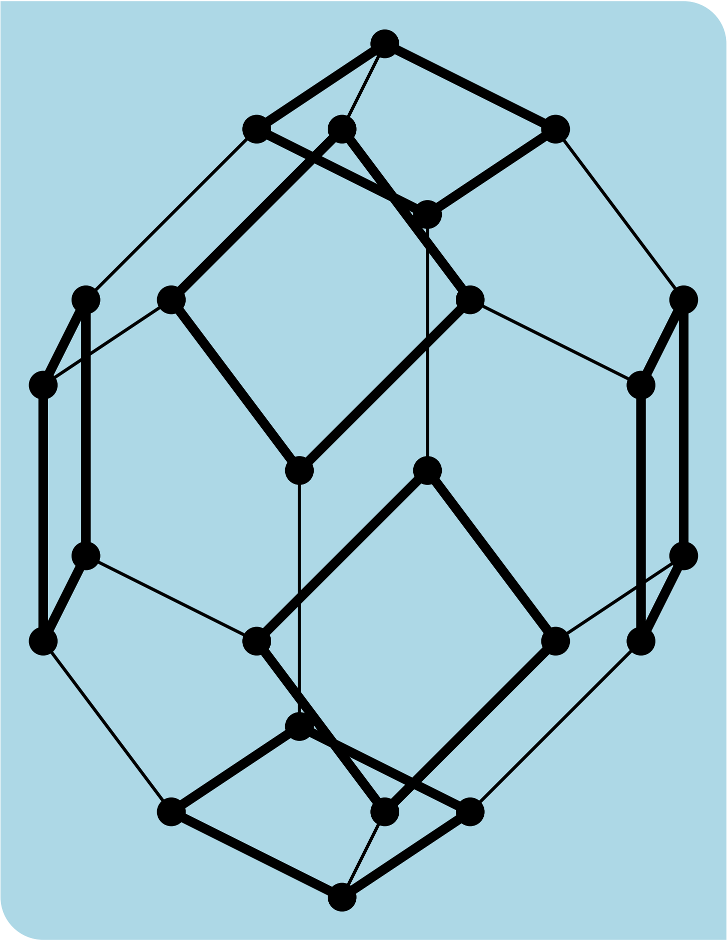

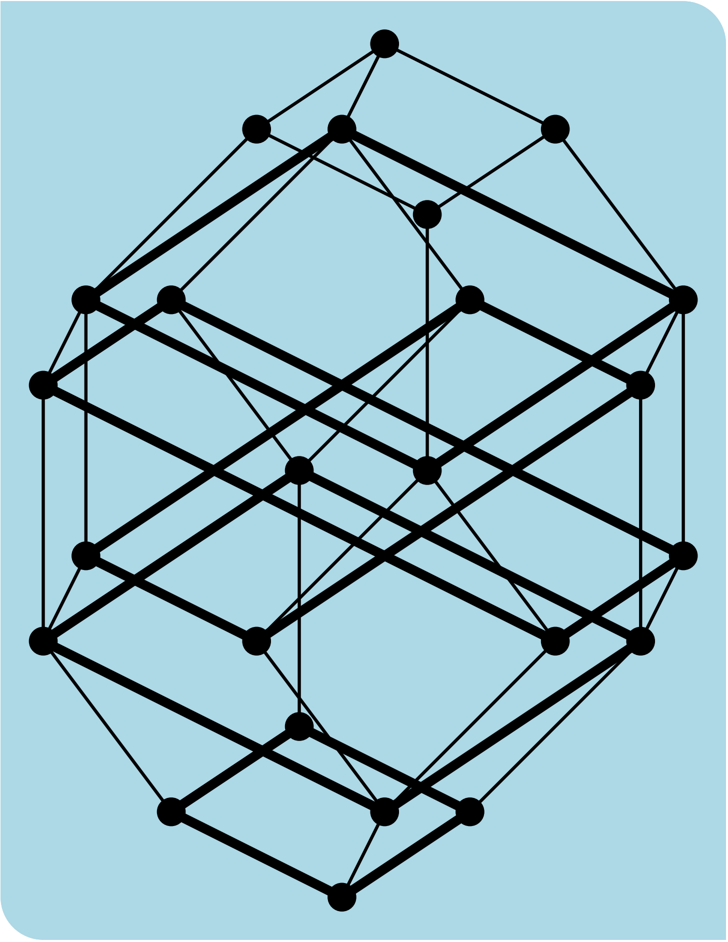

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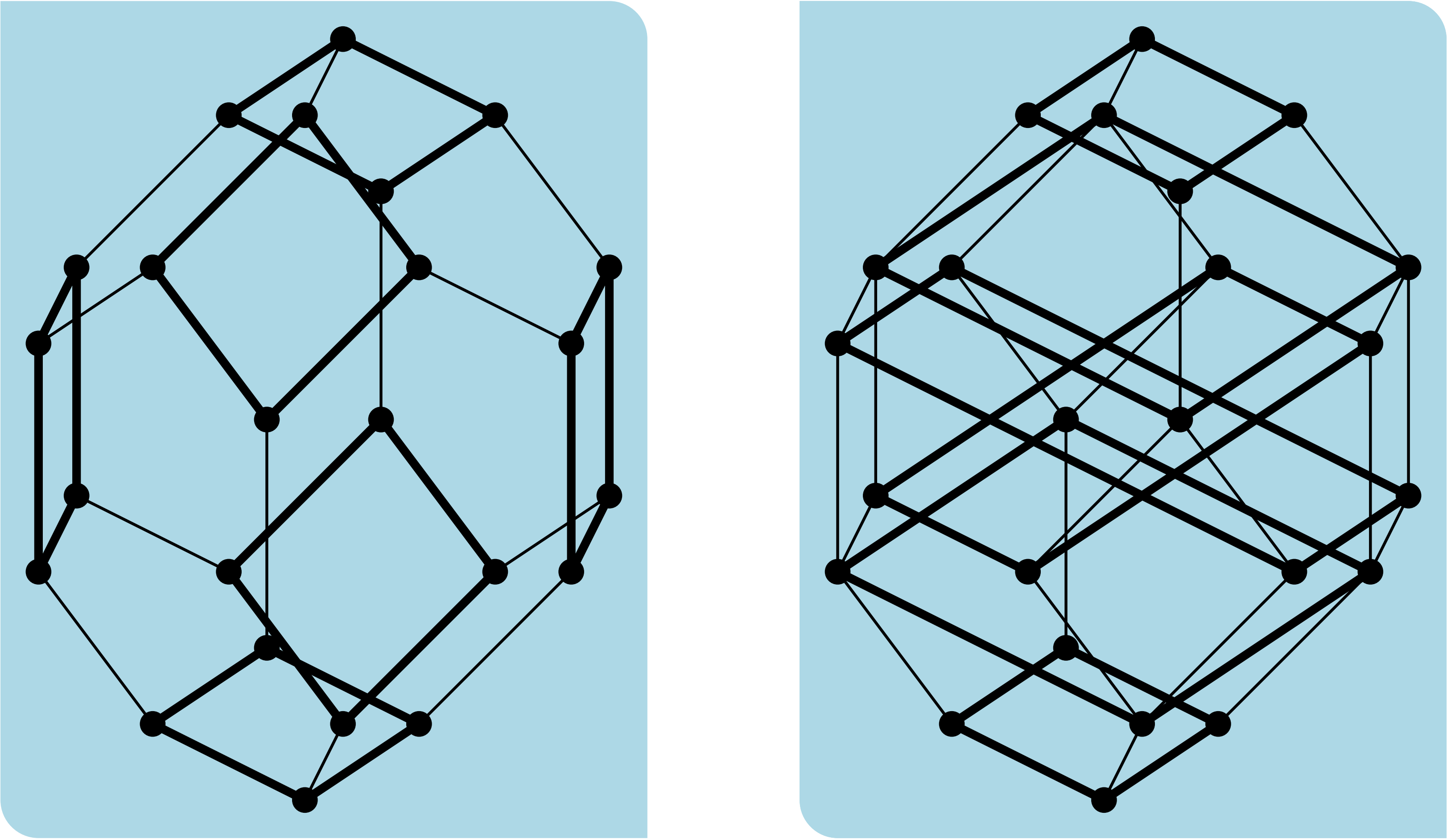

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Open questions

- Are we any closer to proving e-positivity (Stanley–Stembridge 1993)?
- There is a change of base ring for the equivariant cohomology ring in the proof. Is it geometric?
- Lots of possible choices for ζ, but very few land in Sym rather than QSym. Are they special?
- Personal question: does Tymoczko's action live on the variety itself?

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Thank you!