Divisionally free arrangements of hyperplanes

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Setup

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$\mathcal{A} \neq \emptyset$: a central ℓ-arrangement in $V = \mathbb{K}^\ell$.

$H \in \mathcal{A}$.

$\mathcal{A}' := \mathcal{A} \setminus \{H\}$, $\mathcal{A}^H := \{L \cap H \mid L \in \mathcal{A} \setminus \{H\}\}$.

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$\Rightarrow (\mathcal{A}, \mathcal{A}', \mathcal{A}^H)$: the triple.

$L(\mathcal{A}) := \{\cap_{H \in \mathcal{B}} H \mid \mathcal{B} \subset \mathcal{A}\}$: intersection poset.

$L_i(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid \operatorname{codim} X = i\}$.

Localization and restriction

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Two fundamental operations

For $X \in L(\mathcal{A})$, let

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\} \quad \text{(localization)},$$
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Flags

A flag $F = \{X_i\}_{i=0}^{\ell-1}$ of $\mathcal{A}$ is a sequence

$$V = X_0 \supset X_1 \supset \cdots \supset X_{\ell-1}$$

such that $X_i \in L_i(\mathcal{A})$ $(i = 0, \ldots, \ell - 1)$.

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Example

$\mathcal{A}$: arrangement in $\mathbb{R}^4$ defined by

$$\prod_{i=1}^4 x_i \prod_{a_2,a_3,a_4 \in \{\pm 1\}} (x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) = 0.$$

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Then $|\mathcal{A}| = 12$, and a flag is defined, e,g., by

$$\begin{aligned} X_1 = \{x_4 = 0\} &\supset X_2 = \{x_3 = x_4 = 0\} \\ &\supset X_3 = \{x_2 = x_3 = x_4 = 0\}. \end{aligned}$$

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$\mathcal{A}$: the same arrangement

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Then the restrictions are

$$\mathcal{A}^{X_1} : \prod_{i=1}^{3} x_i \prod_{a_2,a_3 \in \{\pm 1\}} (x_1 + a_2x_2 + a_3x_3) = 0,$$

$$\mathcal{A}^{X_2} : x_1x_2(x_1^2 - x_2^2) = 0.$$

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(2) $X_{\ell-1}$ is a line, so $\mathcal{A}^{X_{\ell-1}}$ is a point on the line $X_{\ell-1}$. Hence $|\mathcal{A}^{X_{\ell-1}}| = 1$.

(3) Also, we assume that $X_\ell = \{0\}$ (essential arrangement). Hence $\mathcal{A}^{X_\ell} = \emptyset$, and $|\mathcal{A}^{X_\ell}| = 0$.

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It is known that $\pi(\mathcal{A}; t)$ is combinatorial (i.e., determined by $L(\mathcal{A})$). Hence so are all Betti numbers of the complemeht $M(\mathcal{A}) := \mathbb{C}^\ell \setminus \cup_{H \in \mathcal{A}} H$.

Definition of freeness

Recall the freeness in general.

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Free arrangements

Let $S = \mathbb{K}[x_1, \ldots, x_\ell]$. Then

$$D(\mathcal{A}) := \{\theta \in \operatorname{Der} S \mid \theta(\alpha_H) \in S\alpha_H \ (\forall H \in \mathcal{A})\}.$$

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$$D(\mathcal{A}) := \{\theta \in \operatorname{Der} S \mid \theta(\alpha_H) \in S\alpha_H \ (\forall H \in \mathcal{A})\}.$$

We say $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, d_2, \ldots, d_\ell)$ if

$$D(\mathcal{A}) = S\theta_1 \oplus S\theta_2 \oplus \cdots \oplus S\theta_{\ell-1}$$

with $\deg \theta_i = d_i \ (i = 1, \ldots, \ell)$.

Problems on freeness

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Problems

(1) Are there any relation between freeness (algebraic structure) of $\mathcal{A}$, and $L(\mathcal{A})$ (combinatorial structure) of $\mathcal{A}$?

(2) How to determine freeness of an arrangement?

An answer to Problem1

Factorization Theorem (Terao, 1981)

If $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$, then $\pi(\mathcal{A}; t) = \prod_{i=1}^{\ell}(1 + d_i t)$.

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This is an implication from freeness to combinatorics, and the most important relation between algebra and combinatorics!

An answer to Problem 2

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Addition-Deletion Theorem (Terao, 1980)

For the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}^H)$, any two of the following three imply the third:

(1) $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_{\ell-1}, d_\ell)$.

(2) $\mathcal{A}'$ is free with
$\exp(\mathcal{A}') = (d_1, \ldots, d_{\ell-1}, d_\ell - 1)$.

(3) $\mathcal{A}^H$ is free with $\exp(\mathcal{A}^H) = (d_1, \ldots, d_{\ell-1})$.

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(3) $\mathcal{A}^H$ is free with $\exp(\mathcal{A}^H) = (d_1, \ldots, d_{\ell-1})$.

By Terao's factorization, all the π's above factorize.

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Combining two theorems, we may formulate:

Theorem (Terao, 1980)

$\mathcal{A}$ is free if $\exists H \in \mathcal{A}$ s.t. $\mathcal{A} \setminus \{H\}$ and $\mathcal{A}^H$ are free, and $\pi(\mathcal{A}^H; t)$ divides $\pi(\mathcal{A}; t)$.

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This is the most useful way to determine freeness. The first main theorem in this talk is the following development of the above.

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Compare the addition theorem

Assume that for some $H \in \mathcal{A}$,

(1) $\mathcal{A}^H$ is free, (2) $\pi(\mathcal{A}^H; t) \mid \pi(\mathcal{A}; t)$, and (3) $\mathcal{A} \setminus \{H\}$ is free.

Then $\mathcal{A}$ is free.

Example

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Example : Type B

$\mathcal{B}_\ell$ is defined by $\prod_{i=1}^{\ell} x_i \prod_{1\le i<j\le \ell}(x_i^2 - x_j^2) = 0$. $\mathcal{B}_2$ is free with $\pi(\mathcal{B}_2; t) = (1+t)(1+3t)$,and $\pi(\mathcal{B}_\ell; t) = \prod_{i=1}^{\ell}(1+(2i-1)t)$. Hence division theorem immediately shows that $\mathcal{B}_\ell$ are all free.

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Note that all what we did above are combinatorial, and there are no algebraic arugument, though we are determining freeness!

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All arrangement in $\mathbb{K}^2$ are free, since it coincides with a finite set of lines in $\mathbf{P}^1_{\mathbb{K}}$. Hence every torsion free sheaf on it splits into a direct sum of line bundles.

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Hence applying the division theorem repeatedly, we can obtain a completely combinatorial way to check the freeness!

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Assume that $\mathcal{A}$ has a flag (divisional flag)

$$V = X_0 \supset X_1 \supset \cdots \supset X_{\ell-1}$$

with $X_i \in L_i(\mathcal{A})$ such that $\pi(\mathcal{A}^{X_{i+1}}; t) \mid \pi(\mathcal{A}^{X_i}; t)$ for $i = 0. \ldots, \ell - 2$. Then $\mathcal{A}$ is free.

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\pi(\mathcal{A}^{x_4=0};t) &= (t-1)(t-3)(t-3),\\
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\end{aligned}$$

Hence $\mathcal{A}$ is free with divisional flag
$\mathbb{R}^4 \supset \{x_4 = 0\} \supset \{x_3 = x_4 = 0\}$.

Divisionally free arrangements

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The freeness of $\mathcal{A}$ depends only on $L(\mathcal{A})$, its combinatorics.

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Division theorem and divisional flag work well when we prove Terao's conjecture for several arrangements!

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$$\mathcal{DF} := \cup_{\ell \geq 1} \mathcal{DF}_\ell.$$

Properties of $\mathcal{DF}$; Answer to Problem 2

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Theorem

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Remark

Not all free arrangements are divisionally free! (e.g., the cone of all the edges and diagonals of a regular pentagon.)

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$\mathcal{A} \in \mathcal{IF}$ depends only on combinatorics.

$\mathcal{IF}$ and $\mathcal{DF}$

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In fact, the arrangement $\mathcal{A}(G_{31})$ of the unitary reflection group G_{31} satisfies $\mathcal{A}(G_{31}) \in \mathcal{DF} \setminus \mathcal{IF}$ due to the result by Röhrle and Hoge.

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The second Betti number

Let $b_2(\mathcal{A})$ denote the second Betti number of $M(\mathcal{A}) := \mathbb{C}^\ell \setminus \cup_{H\in\mathcal{A}} H$ when $\mathbb{K} = \mathbb{C}$. In fact,

$$b_2(\mathcal{A}) = \sum_{X \in L_2(\mathcal{A})} (|\mathcal{A}_X| - 1)$$

over an arbitrary field $\mathbb{K}$ by Orlik-Solomon.

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$$\pi(\mathcal{A};t) = \prod_{i=0}^{\ell-1}(1 + (|\mathcal{A}^{X_i}| - |\mathcal{A}^{X_{i+1}}|)t).$$

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$\pi(\mathcal{A}; t) = \prod_{i=0}^{\ell-1}(1 + (|\mathcal{A}^{X_i}| - |\mathcal{A}^{X_{i+1}}|)t)$.

(3) $\exists \{X_i\}$ a flag s.t.

$$b_2(\mathcal{A}) = \sum_{i=0}^{\ell-2}(|\mathcal{A}^{X_i}| - |\mathcal{A}^{X_{i+1}}|)|\mathcal{A}^{X_{i+1}}|.$$

(b_1,b_2)-inequality

In particular, we can show that

(b_1, b_2)-inequality

$$
\begin{aligned}
b_2(\mathcal{A}) &\geq \sum_{i=0}^{\ell-2} (|\mathcal{A}^{X_i}| - |\mathcal{A}^{X_{i+1}}|)|\mathcal{A}^{X_{i+1}}| \\
&= \sum_{i=0}^{\ell-2} (b_1(\mathcal{A}^{X_i}) - b_1(\mathcal{A}^{X_{i+1}}))b_1(\mathcal{A}^{X_{i+1}})
\end{aligned}
$$

for any flag $\{X_i\}$, and the equality holds if and only if $\mathcal{A} \in \mathcal{DF}$.

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$\mathcal{A}$:

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Take a flag defined by

$$X_1 = \{x_4 = 0\} \supset X_2 = \{x_3 = x_4 = 0\}$$
$$\supset X_3 = \{x_2 = x_3 = x_4 = 0\}.$$

Example again 2

Compute b_2 and b_1's!

Then

$$\begin{aligned} b_2(\mathcal{A}) &= 50,\ |\mathcal{A}| = 12, \\ |\mathcal{A}^{X_1}| &= 7,\ |\mathcal{A}^{X_2}| = 4,\ |\mathcal{A}^{X_3}| = 1. \end{aligned}$$

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Hence

$$50 = (12-7)7 + (7-4)4 + (4-1)1$$

confirms that $\mathcal{A} \in \mathcal{DF}$.

More applications

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Applications of divisions

(1) Combinatoriality of most of recursively free arrangements.

(2) Combinatorial freeness of Coxeter and unitary reclection arrangements and its relatives.

Outline of the proof of division theorem

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The proof depends on algebraic geometry (Horrocks' splitting criterion) and multiarrangement theory.

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Let $T_{\mathcal{A}} := \widetilde{D_0(\mathcal{A})}$ and take $H \in \mathcal{A}$. Then $\mathcal{A}$ is free iff $T_{\mathcal{A}}$ splits iff $T_{\mathcal{A}}|_H$ splits by Horrocks. Hence for the division, we need to approximate $T_{\mathcal{A}}|_H$ in terms of $\mathcal{A}^H$!

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Remark

For b_2-version, we use Poincarè polynomial of multiarrangement by Terao, Wakefield and myself. That is close to Chern polynomial of $T_{\mathcal{A}}|_H$ by Schulze, A-Yoshinaga and Denham-Schulze. i.p., $b_2(\mathcal{A}) = c_2(T_{\mathcal{A}})$ by Denham-Schulze.

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Let $H \neq L \in \mathcal{A}$, $X := H \cap L \in L_2(\mathcal{A})$.Assume that $\mathcal{A}$ and $\mathcal{A}^X$ are free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$, $\exp(\mathcal{A}^X) = (d_1, \ldots, d_{\ell-2})$.

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We may play with this in type D_ℓ arrangement.

Sandwich Theorem : Example

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Sandwich example

$\mathscr{A}$: free arrangement with $\exp(\mathscr{A}) = (1, 3, 3, 5)$ defined by

$\prod_{i=1}^{4} x_i \prod_{a_2,a_3,a_4 \in \{\pm 1\}} (x_1 + a_2x_2 + a_3x_3 + a_4x_4) = 0.$

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Take a different flag

$$\begin{aligned} Y_1 &= \{x_1 + x_2 + x_3 - x_4 = 0\} \\ &\supset Y_2 := Y_1 \cap \{x_1 - x_2 + x_3 + x_4 = 0\}. \end{aligned}$$

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Then is $\mathcal{A}^{Y_1}$ free?

Sandwich example

Compute only b_1's!

Then

$$\begin{aligned} \exp(\mathcal{A}) &= (1,3,3,5),\ |\mathcal{A}| = 12,\ |\mathcal{A}^{Y_1}| = 7, \\ |\mathcal{A}^{Y_2}| &= 4 \Rightarrow \exp(\mathcal{A}^{Y_2}) = (1,3). \end{aligned}$$

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Since $|\mathcal{A}| - |\mathcal{A}^{Y_1}| = 5 \in \exp(\mathcal{A}) \setminus \exp(\mathcal{A}^{Y_2})$, Sandwich theorem shows $\mathcal{A}^{Y_1}$ is free with $\exp(\mathcal{A}^{Y_1}) = (1,3,3)$.

DF and SS

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New(?) characterization of supersolvable arrangement

$\mathcal{A}$ is supersolvable if and only if $\exists \{X_i\}$ a flag s.t.

$$b_2(\mathcal{A}) = \sum_{i=0}^{\ell-2} (|\mathcal{A}_{X_{i+2}}| - |\mathcal{A}_{X_{i+1}}|)|\mathcal{A}_{X_{i+1}}|.$$

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In this case, $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (|\mathcal{A}_{X_\ell}| - |\mathcal{A}_{X_{\ell-1}}|, |\mathcal{A}_{X_{\ell-1}}| - |\mathcal{A}_{X_{\ell-2}}|, \ldots, |\mathcal{A}_{X_1}|)$.

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$\exists$Similarity between SS and DF?

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(2) Division Theorem asserts that $\mathcal{A}$ is free if $\mathcal{A}^H$ is free with a combinatorial condition. How about the converse? (A modification of Orlik's conjecture, Sandwich theorem).

(3) Does the similar statement to division and its flag hold true for other arrangements or divisors?

Thanks

Thank you for your attention!