Symmetries of the roots of b-functions: a survey

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§0.- Introduction

Algebraic setting : $k=\mathbb{C}$ or $k=$ a field of ch. $=0$

$R=\mathcal{O}_{\mathbb{C}^n,0}$, $k[x_1,\dots,x_n]$, $k[[\underline{x}]]$, "diff. admissible k-algebras"

Main differential objects: $\mathrm{Der}_k(R)$, $\mathcal{D}=\mathcal{D}_{R/k}$

BS-polynomials or b-functions : $h\in R \rightsquigarrow$

$\exists\, b(s)\in k[s]$, $b(s)\neq 0$, $\exists\, P(s)\in\mathcal{D}[s]$ s.t.

$$b(s)\,h^s = P(s)\,h^{s+1}$$

(Bernstein, Kashiwara, Björk, Mebkhout-NM, Núñez-Betancourt)

$$b_h(s) := \text{minimal polynomial of } s \circlearrowright \frac{\mathcal{D}[s]h^s}{\mathcal{D}[s]h^{s+1}}$$

The reduced b-function:

$$\tilde{b}_h(s)=\frac{b_h(s)}{s+1} = \text{min. pol. of } \quad s \circlearrowright \frac{\mathcal{D}[s]\,h^s}{\mathcal{D}[s]\,J\,h^s}$$

The talk is concerned with symmetry properties of the following type: $\tilde{b}(s) = \pm \tilde{b}(-s-e)$, e fixed.

or: $b(s) = \pm b(-s-2)$

Motivation:

-) Granger-Schulze 2010: reductive prehomogeneous determinants, special linear free divisors

-) Isolated q-h singularities

-) b-functions associated with ILC with respect to q-h plane curves.

§1.- The free case

REDUCED

<u>Definition</u>: $h \in R$ is free (Saito) if

$\mathrm{Der}(-\log h) = \{\delta \in \mathrm{Der}_k(R) \mid \delta(h) \in \langle h \rangle\}$

is a R-free module (of rank $= n$)

Assume h (reduced) free: $\delta_i(h) = \alpha_i h$, $1 \leq i \leq n$

$\delta_i = \sum a_{ij} \partial_j$; Saito's crit.: $\det(a_{ij}) = (\text{unit})\, h$

*) Logarithmic Bernstein construction (Calderón-NM, 2009)

$$\mathcal{V} = R[\partial_1, \ldots, \partial_n] \subset \mathcal{D}$$

$R[s]\, h^s$

$R[s]\, h^s = \mathcal{V} / \mathcal{V}\langle \delta_i - \alpha_i s \rangle$

One can construct a Spencer resolution of $R[s]h^s$ over $\mathcal{V}[s]$.

$$R[s]h^s \subset R[s, h^{-1}]h^s$$

Everything make sense for $h^{q(s)}$, $q(s) \in k[s]$.

There is a comparaison map:

$$\textcircled{\star} \quad \mathcal{D}[s] \overset{\mathbb{L}}{\underset{\mathcal{V}[s]}{\otimes}} R[s]h^s \longrightarrow \mathcal{D}[s]h^s \left(\subset R[s,h^{-1}]h^s \right)$$

The left side term can be computed by means of the logarithmic Spencer resolution.

<u>Theorem</u>: If $\textcircled{\star}$ is an isomorphism, then:

$$\mathbb{D}\left(\mathcal{D}[s]h^s\right) = \mathcal{D}[s]h^{-s-1}$$

This is a consequence of a duality formula of Calderón–NM (2005) relative to an extension of Lie-Rinehart algebras:

$$\mathrm{Der}(-\log h) \subset \mathrm{Der}_k(R)$$

$$\mathcal{V} = U(\mathrm{Der}(-\log h)) \subset \mathcal{D} = U(\mathrm{Der}_k(R))$$

$$\mathbb{D}_{\mathcal{D}[s]}\left(\mathcal{D}[s] \overset{L}{\underset{\mathcal{V}[s]}{\otimes}} \mathcal{E}\right) \simeq \mathcal{D}[s] \overset{L}{\underset{\mathcal{V}[s]}{\otimes}} \mathbb{D}_{\mathcal{V}}(\mathcal{E})\, h^{-1}$$

$$\mathbb{D}_{\mathcal{V}}(R[s]\, h^{s}) = R[s]\, h^{-s}$$

We have to take $\mathbb{D}_{\mathcal{D}[s]}$ on:

$$0 \to \mathcal{D}[s] h^{s+1} \longrightarrow \mathcal{D}[s]\, h^{s} \longrightarrow Q(s) = \frac{\mathcal{D}[s] h^{s}}{\mathcal{D}[s] h^{s+1}} \longrightarrow 0$$

$$\mathbb{D}(Q(s)) = Q(-s-2)\,[-1]$$

$$b_h(s) = \pm\, b_h(-s-2)$$

Sufficient conditions for (★) to be $\simeq$

a) $\mathcal{D}[s] \otimes_{\mathcal{V}[s]} R[s]h^s \simeq \mathcal{D}[s]\,h^s \iff \operatorname{ann}_{\mathcal{D}[s]} h^s$ is generated by order 1 op.

b) $\mathcal{D}[s] \overset{\mathbb{L}}{\otimes}_{\mathcal{V}[s]} R[s]h^s$ is concentrated in $\deg = 0$

h "linear Jacobian type" $\Rightarrow$ a)

h "linear Jacobian type" $\Rightarrow \{h, \sigma(\delta_1 - \alpha_1 s), \dots, \sigma(\delta_n - \alpha_n s)\}$ is a regular sequence in $\operatorname{gr}(\mathcal{D})[s] \Rightarrow$ b)

Free hyperplane arrangements $\Rightarrow$ Free loc. q-h $\Rightarrow$ Free linear Jacobian

§2.- Q-H isolated singularities revisited (with Arcadias)

$$\eta = \overline{h^s} \in \frac{\mathcal{D}[s]h^s}{\mathcal{D}[s]Jh^s} =: \widetilde{Q}(s)$$

$$\operatorname{ann}_{\mathcal{D}[s]} \eta = \mathcal{D}[s]\langle h'_{x_i}, h'_{x_j}\partial_i - h'_{x_i}\partial_j, \chi - s\rangle =$$

$$= \mathcal{D}[s]\langle h'_{x_1}, \ldots, h'_{x_n}, \chi - s\rangle$$

We can form a Spencer complex again and:

$$\mathbb{D}(\widetilde{Q}(s)) = \widetilde{Q}(-s-n)[-1]$$

§3.- More symmetry properties

*) Thom-Sebastiani join

$f(x):(\mathbb{C}^m,0)\to(\mathbb{C},0)$, $g(y):(\mathbb{C}^n,0)\to(\mathbb{C},0)$

$$h=f(x)+g(y):(\mathbb{C}^{m+n},0)\longrightarrow(\mathbb{C},0)$$

Assume $f\in\langle f'_{x_1},\dots,f'_{x_m}\rangle$

<u>Theorem</u> (M. Saito, 1994):

$$\tilde{b}_{f+g}=\tilde{b}_f*\tilde{b}_g$$

With: $\alpha*\beta$ defined as:

$$\underbrace{\langle\alpha(t),\beta(s-t)\rangle}_{\substack{\cap\\ k[s,t]}}\cap k[s]=\langle(\alpha*\beta)(s)\rangle$$

A purely algebraic proof is still missing

If: $\alpha(s) = \pm\alpha(-s-d)$, $\beta(s) = \pm\beta(-s-e) \Rightarrow$
$\Rightarrow (\alpha * \beta)(s) = \pm(\alpha * \beta)(-s-(d+e))$

In that way we can produce examples of h with $\tilde{b}_h(s) = \pm\tilde{b}_h(-s-e)$, $1 \leq e \leq n$, satisfying:

-) J linear type

-) $\mathcal{O}/J$ C.M.

-) $e = \operatorname{codim}\{h=0\}^{\text{sing}}$

-) even irreducible

That suggests a Conjecture . . .

§4.– Looking for a general setting

Quasi-free structures (Damon, Castro-Ucha) are in principle adapted to study:

-) free multiarrangements
-) prehomogeneous determinants
-) Some nearly free divisors (Dimca)

This is work in progress in collab. with F. Castro, D. Mond.

The basic idea is that we can start with a rank n free Lie-Rinehart algebra $L \subset \mathrm{Der}_k(R)$

In this case: h = det. of Saito's matrix of L

Condition ✪ should be relaxed. For instance, some examples suggest the following:

✪' $\exists\, \beta(s) \in k[s]$, $\beta(s) \neq 0$, s.t. condition ✪ becomes true after inverting $\beta(s)$:

$$\left(\mathcal{D}[s] \overset{\mathbb{L}}{\underset{\mathcal{V}[s]}{\otimes}} R[s] h^s\right)_{\beta(s)} \xrightarrow{\sim} \left(\mathcal{D}[s] h^s\right)_{\beta(s)}$$

$$\mathcal{D}[s, 1/\beta] \overset{\mathbb{L}}{\underset{\mathcal{V}[s,1/\beta]}{\otimes}} R[s, 1/\beta] h^s \xrightarrow{\sim} \mathcal{D}[s, 1/\beta] h^s \left(\subset R[s, 1/h, 1/\beta] h^s\right)$$

This property would give the symmetry of the b-function up to factors of $\text{l.c.m.}(\beta(s), \beta(-s-2)) = \tilde{\beta}(s)$.

From here we can deduce the logarithmic comparaison theorem whenever $\tilde{\beta}(l) \neq 0$ for all integers $l < -1$.