Cutting, embedding, bouncing characteristic classes

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based on joint work with Imre Bárány, Frederick Cohen, Wolfgang Lück, Roman Karasev, András Szűch and Günter M. Ziegler

supported by Anthony Bahri, Martin Bendersky, Peter Landweber, and John McCleary

Cutting

Conjecture (Nandakumar & Ramana Rao 2006)

Every convex polygon P in the plane can be partitioned into any prescribed number k of convex pieces that have equal area and equal perimeter.

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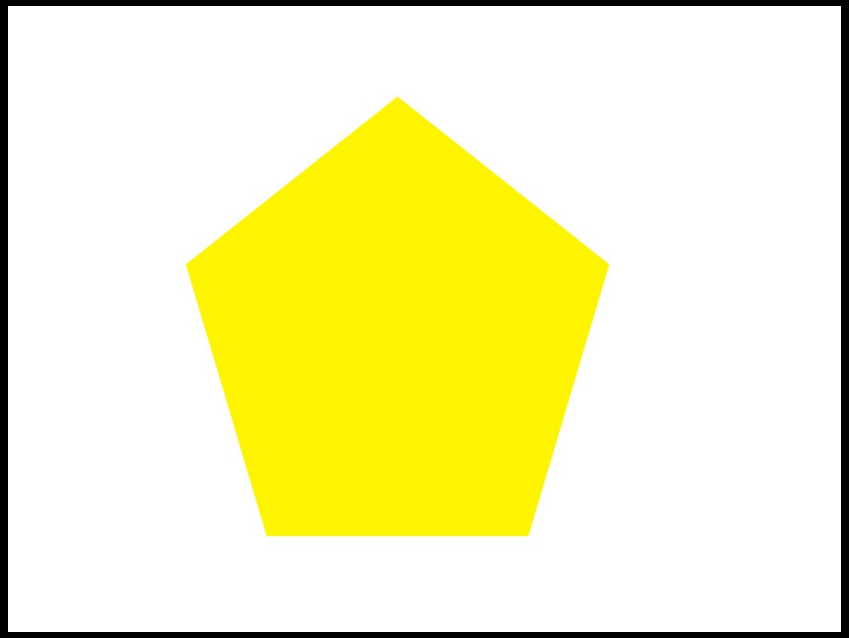

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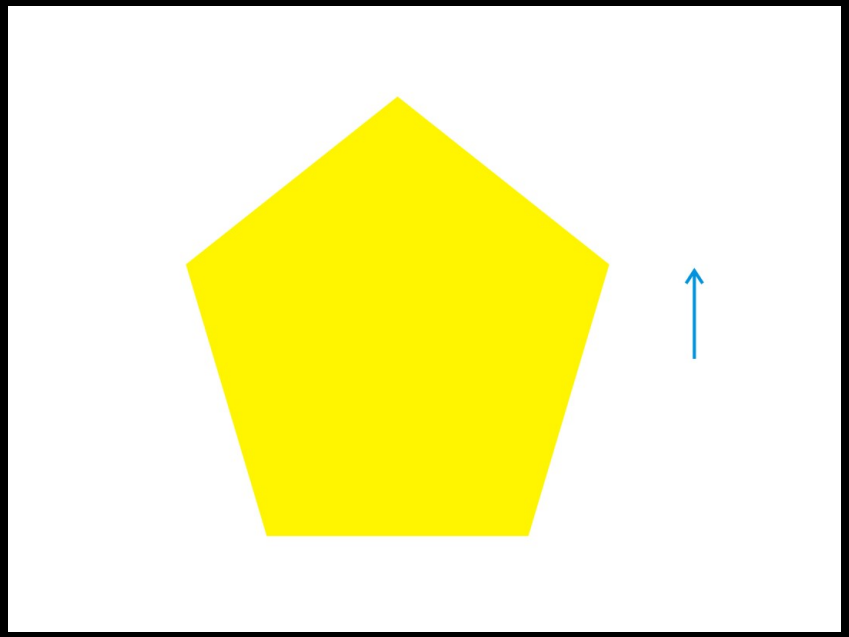

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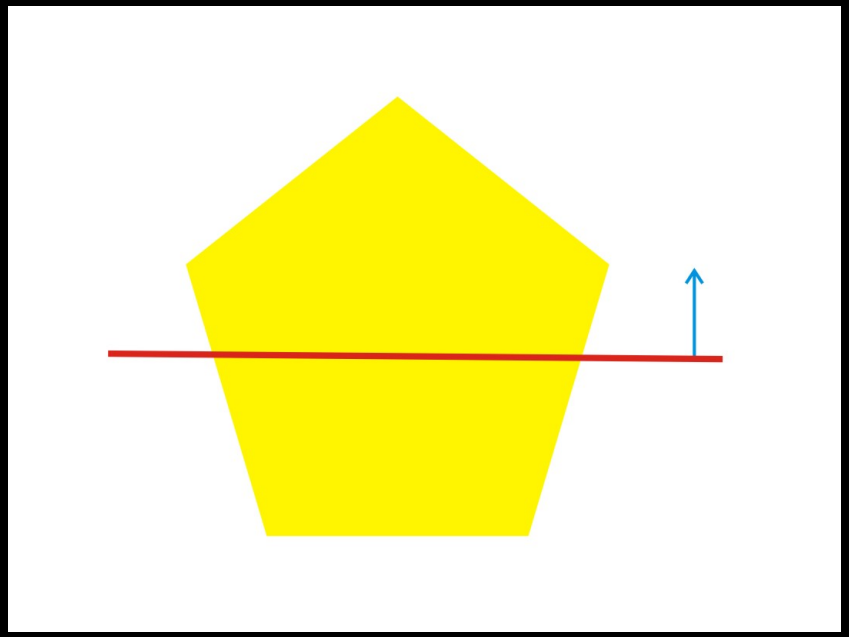

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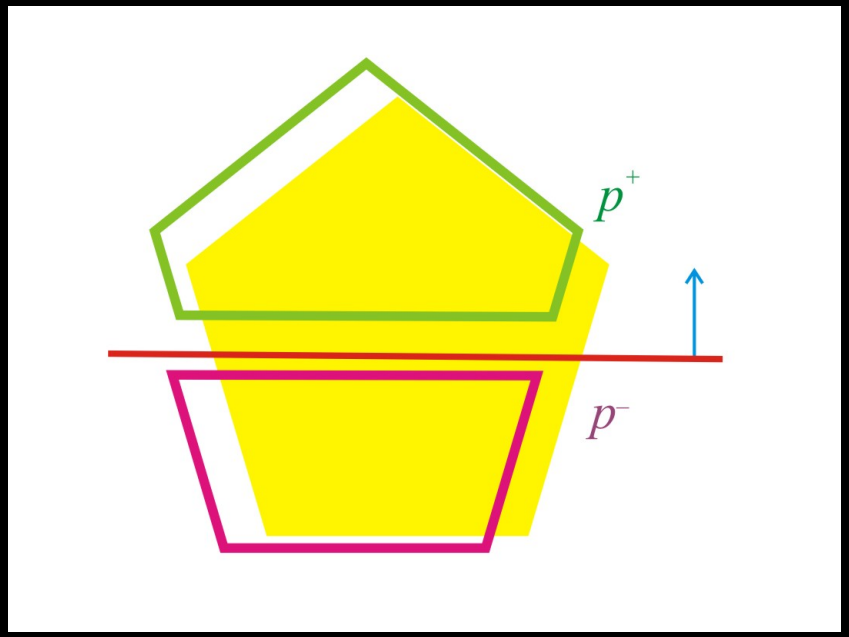

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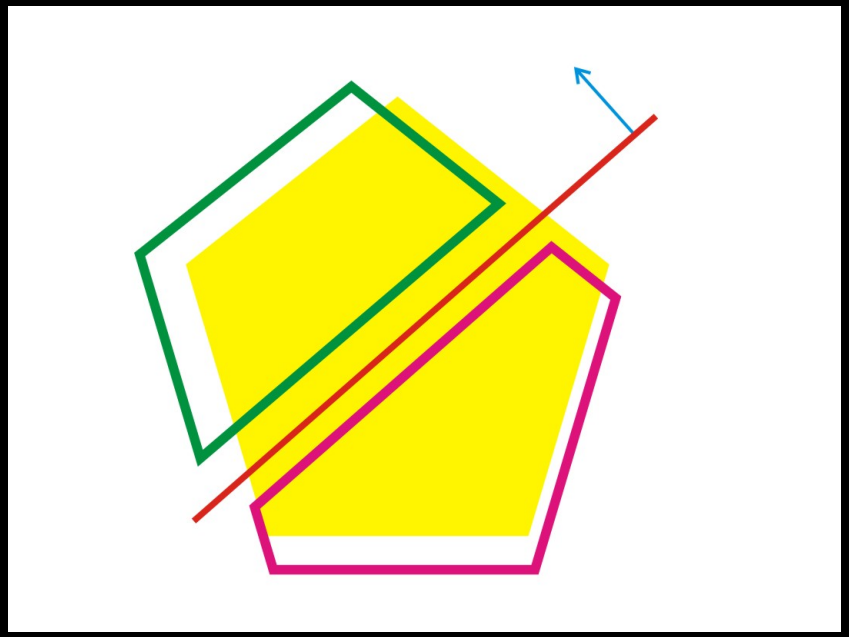

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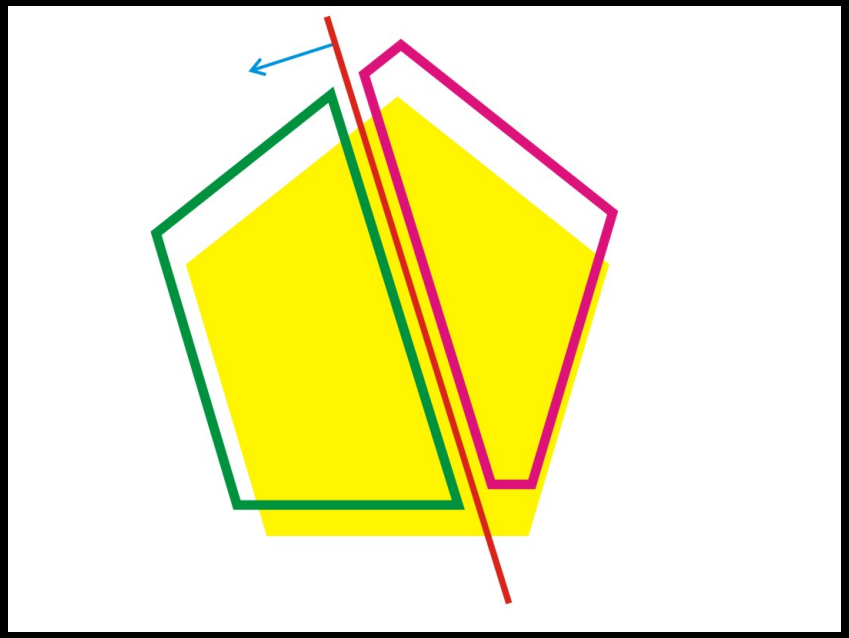

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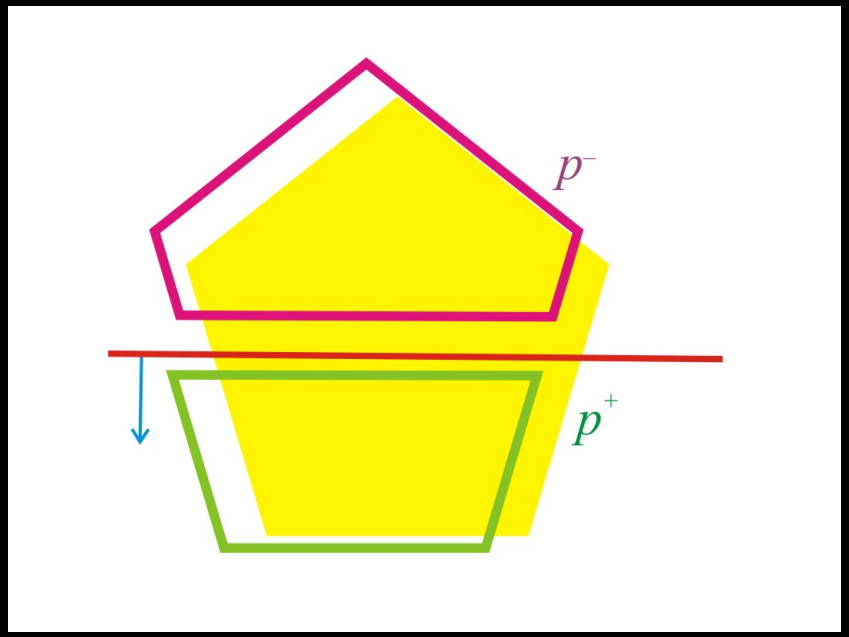

Cutting

Missbrauch melden Nächstes Blog» Blog erstellen Anmelden

TECH-MUSINGS

Thoughts On Algorithms, Geometry etc...

THURSDAY, SEPTEMBER 28, 2006

'Fair' Partitions

Let us (N. Ramana Rao and self) float yet another geometric conjecture

****

Given any convex shape and any positive integer N. There exist some way(s) of partitioning this shape into N convex pieces so that all pieces have equal area and equal perimeter.

****

Note: We could define the 'fair partition' of a polygon as a partition into pieces of equal area and perimeter and further, a 'convex fair partition' as a fair partition in which the pieces are also convex. The claim above could be restated as: "for any N, any convex shape allows convex fair partitioning into

About Me

Name: R.Nandakumar
Location: India

Computer Programmer, Student of Mathematics, General Dabbler.

View my complete profile

Previous Posts

Traversing A Tree
A Sequence Puzzle

Cutting

I could not find any known result on this - although there has been work on dividing into pieces so that every piece has same area and the same fraction of the outer boundary of the target. The requirement that all pieces should have same perimeter is not very easy to ensure - at least not as easy as ensuring equal areas for them. One does not know ab initio what precise perimeter each piece should have!

posted by R.Nandakumar @ 6:57 AM 0 commen

Embedding

Definition

A continuous map $f : X \to \mathbb{R}^N$ $(\mathbb{C}^N)$ ***is*** k **-regular** ***if for every pairwise distinct points*** $x_1, \ldots, x_k$ ***the vectors***

$$f(x_1), \ldots, f(x_k) \text{ are linearly independent.}$$

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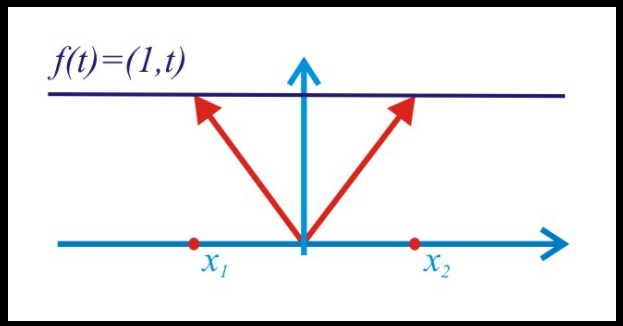
Example

(1) $f : \mathbb{R} \to \mathbb{R}^k$, $f(t) = (1, t, \ldots, t^{k-1})$ ***is a*** k***-regular map.***



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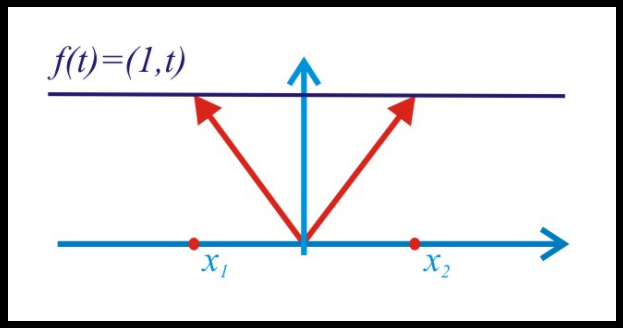

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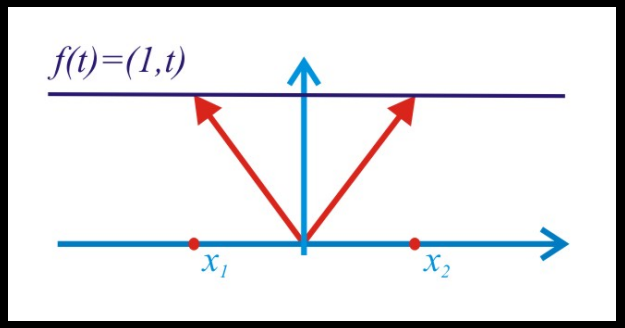

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(3) $f : \mathbb{C} \to \mathbb{C}^k$, $f(z) = (1, z, \ldots, z^{k-1})$ *is complex k-regular.*

Conjecture (Cohen & Handel 1978, Chisholm 1979)

If $\mathbb{R}^d \to \mathbb{R}^N$ is k-regular, then $N \geq d(k - \alpha_2(k)) + \alpha_2(k)$.

Further embeddings

Definition

Affine subspaces $L_1, \ldots, L_\ell$ ***of*** $\mathbb{R}^N$ ***are affinely independent if***

$$\dim \operatorname{aff}(L_1 \cup \cdots \cup L_\ell) = (\dim L_1 + 1) + \cdots + (\dim L_\ell + 1) - 1.$$

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Definition

A smooth map $f\colon M \to \mathbb{R}^N$ is ℓ-skew if pairwise distinct points $y_1, \ldots, y_\ell$ on M the affine subspaces

$$(\iota \circ df_{y_1})(T_{y_1}M), \ldots, (\iota \circ df_{y_\ell})(T_{y_\ell}M)$$

are affinely independent.

Here $df\colon TM \to T\mathbb{R}^N$ and $\iota\colon T\mathbb{R}^N \to \mathbb{R}^N$, $(x, v) \mapsto x + v$.

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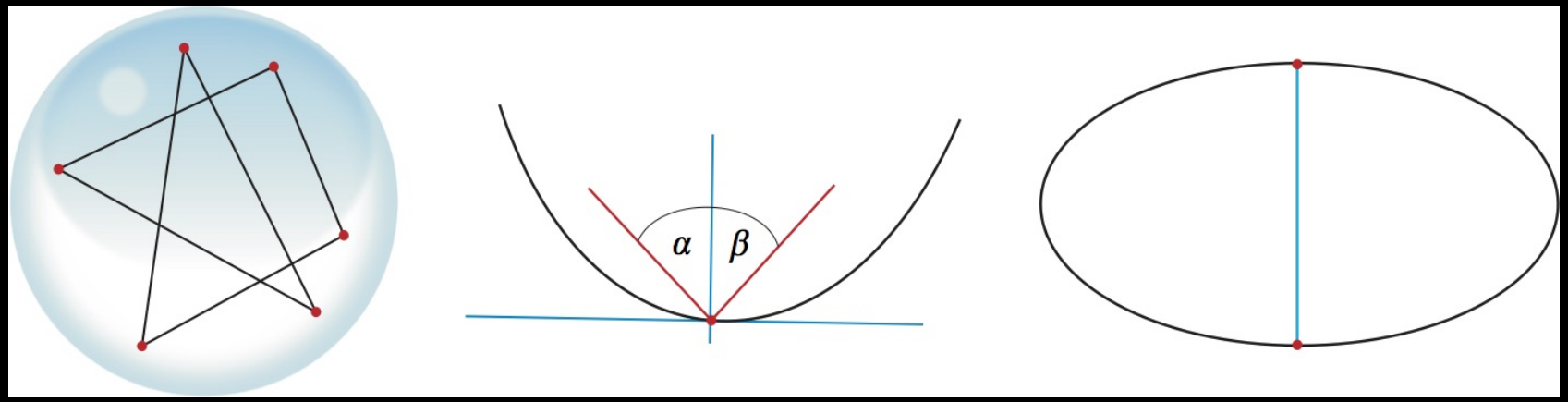
Conjecture

If $\mathbb{R}^d \to \mathbb{R}^N$ is ℓ-skew, then

$$N \geq 2^{\gamma(d)}(\ell - \alpha_2(\ell)) + (d+1)\alpha_2(\ell)$$

where $\gamma(d) = \lfloor \log_2 d \rfloor + 1$.

Bouncing



A billiard trajectory of *length* k in a smooth strictly convex body $T^d \subset \mathbb{R}^d$ is a sequence $(x_0, \ldots, x_k)$ of points on ∂T^d such that for every $0 < i < k+1$ the interior normal to ∂T^d at the point x_i bisects the angle $\angle x_{i-1} x_i x_{i+1}$.

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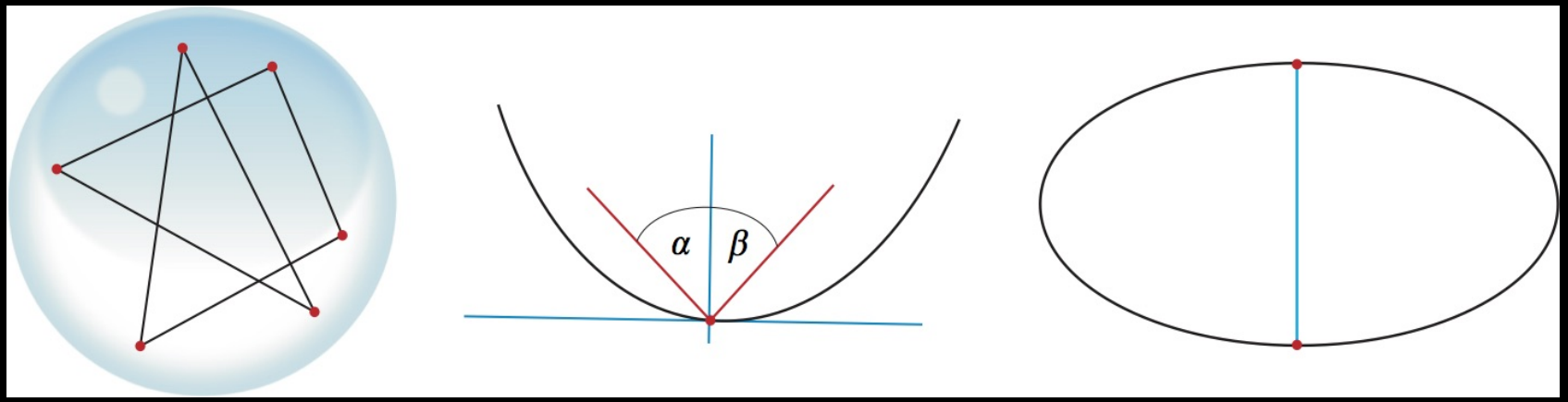

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If $x_0 = x_k$ then the billiard trajectory $(x_0, \ldots, x_k)$ is called periodic billiard trajectory of *period* n if the normal to ∂T^d at the point x_0 bisects the angle $\angle x_{k-1} x_0 x_1$.

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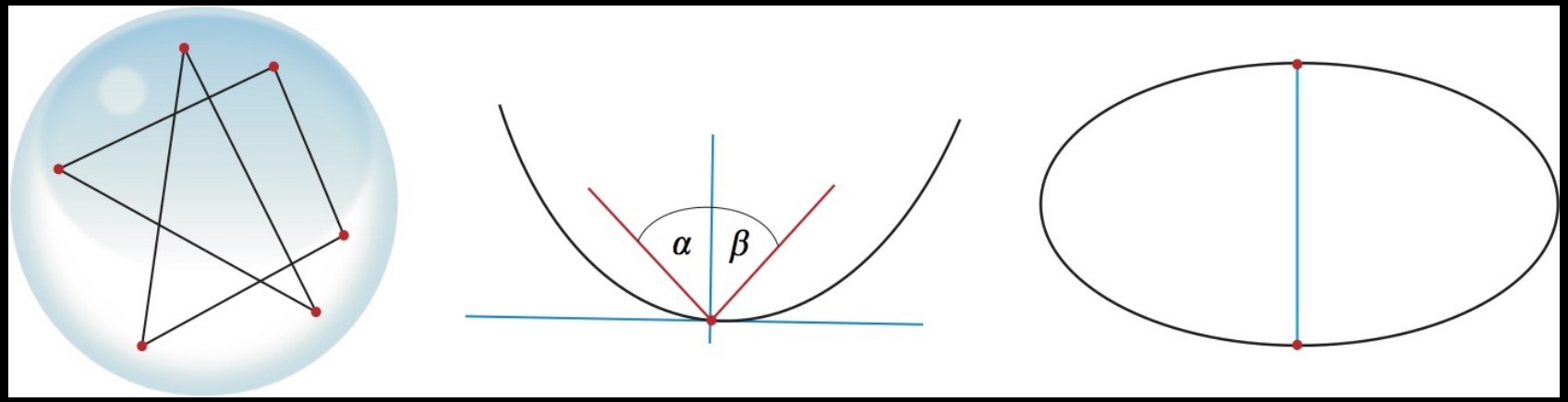

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Problem

Let and $T^d \subset \mathbb{R}^d$ be a smooth strictly convex body. Estimate the number of periodic billiard trajectories $N(T^d, k)$ of the period k on the body T^d modulo the action of dihedral group D_{2k}.

The vector bundle

- $\mathbb{R}^k$ **is a** $\mathfrak{S}_k$**-representation, where** $\mathfrak{S}_k$ **permutes coordinates**
- $W_k := \{(x_1, \ldots, x_k) \in \mathbb{R}^k : \sum_{i=1}^k x_i = 0\}$ **a subrepresentation**
- $T_k := \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_1 = \cdots = x_k\}$ **a subrepresentation**

$$\mathbb{R}^k \cong W_k \oplus T_k$$

$$\begin{tikzcd} \mathbb{R}^k \arrow[r] & F(X,k) \times \mathbb{R}^k \arrow[d] \\ & F(X,k) \end{tikzcd}$$

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$$\begin{tikzcd}
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The vector bundle

$$\xi_{X,k}: \qquad \mathbb{R}^k \longrightarrow F(X,k) \times_{\mathfrak{S}_k} \mathbb{R}^k \longrightarrow F(X,k)/\mathfrak{S}_k$$

$$\zeta_{X,k}: \qquad W_k \longrightarrow F(X,k) \times_{\mathfrak{S}_k} W_k \longrightarrow F(X,k)/\mathfrak{S}_k$$

$$\tau_{X,k}: \qquad T_k \longrightarrow F(X,k)/\mathfrak{S}_k \times T_k \longrightarrow F(X,k)/\mathfrak{S}_k$$

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$$\begin{aligned} w(\xi_{X,k}) &= w(\zeta_{X,k}) \cdot w(\tau_{X,k}) = w(\zeta_{X,k}) \\ c(\xi_{X,k} \otimes \mathbb{C}) &= c(\zeta_{X,k} \otimes \mathbb{C}) \cdot c(\tau_{X,k} \otimes \mathbb{C}) = c(\zeta_{X,k} \otimes \mathbb{C}) \end{aligned}$$

The vector bundle

$$\xi_{X,k}: \qquad \mathbb{R}^k \longrightarrow F(X,k) \times_{\mathfrak{S}_k} \mathbb{R}^k \longrightarrow F(X,k)/\mathfrak{S}_k$$

- $\xi_{\mathbb{R}^d,2}$ **is stably isomorphic to the canonical bundle over** $\mathbb{R}\mathrm{P}^{d-1}$
- **F. Cohen, R. Cohen, Kuhn, Neisendorfer:**

  Bundles over configuration spaces,

  Pacific J. of Math., 1983

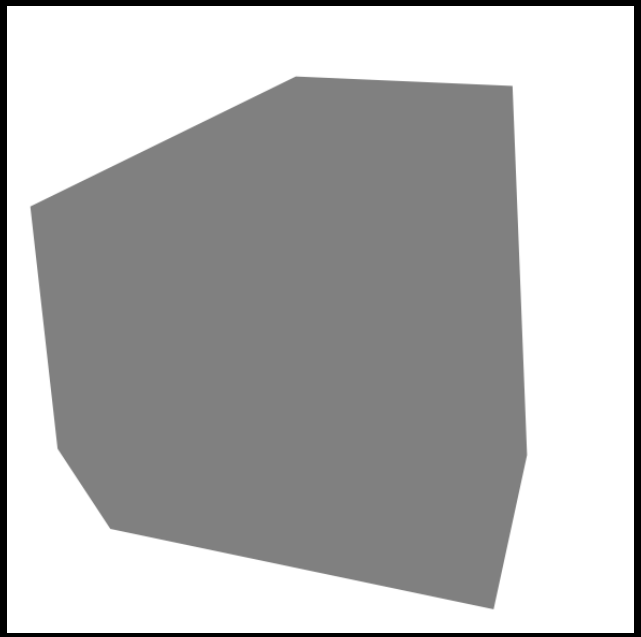
  Stable order of $\xi_{\mathbb{R}^d,k}$ = **minimal** m **such that** $m\xi_{\mathbb{R}^d,k}$ **stabely trivial**

  $= 2^{\rho(d-1)} \prod_{3 \le p \le k} p^{\lfloor \frac{d-1}{2} \rfloor}$ **for** $d \neq 0 \mod 4$

Connection

cutting	$e(\zeta_{\mathbb{R}^d,k}^{\oplus d-1}, \mathcal{Z}^{\otimes d-1})$	$H^*(F(\mathbb{R}^d,k)/\mathfrak{S}_k; \mathcal{Z}^{\otimes d-1})$
k**-regular**	$\bar{w}_{(d-1)(k-\alpha_2(k))}(\xi_{\mathbb{R}^d,k})$	$H^*(F(\mathbb{R}^d,k)/\mathfrak{S}_k; \mathbb{F}_2)$
ℓ**-skew**	$\bar{w}_{(2^{\gamma(d)}-d-1)(\ell-\alpha_2(\ell))}(\xi_{\mathbb{R}^d,\ell}^{\oplus d+1})$	$H^*(F(\mathbb{R}^d,k)/\mathfrak{S}_k; \mathbb{F}_2)$
comp. k**-regular**	$\bar{c}_{(d-1)(k-\alpha_p(k))}(\xi_{\mathbb{C}^d,k} \otimes \mathbb{C})$	$H^*(F(\mathbb{R}^d,k)/\mathfrak{S}_k; \mathbb{F}_p)$
bouncing	$e(\zeta_{\mathbb{R}^d,p}^{\oplus d-1}, \mathcal{Z}^{\otimes d-1})$	$H^*(F(\mathbb{R}^d,p)/\mathfrak{S}_p; \mathcal{Z}^{\otimes d-1})$

Cutting polygons and the twisted Euler class of $\zeta_{\mathbb{R}^d,k}^{\oplus d-1}$



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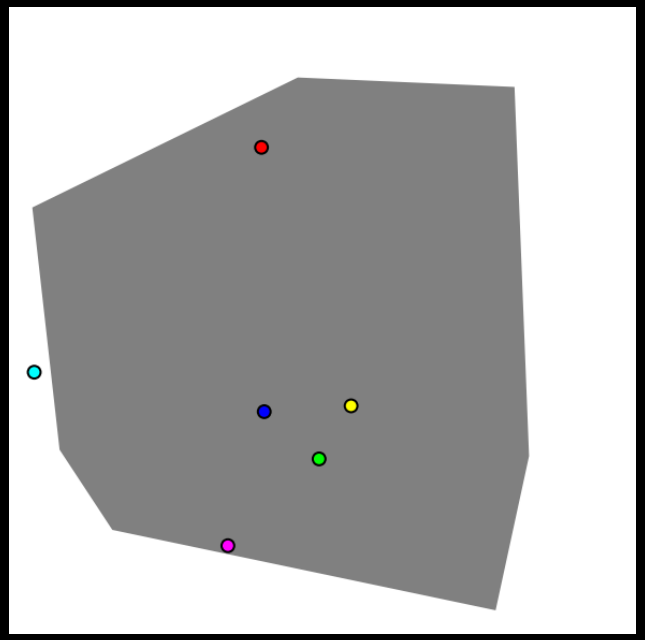

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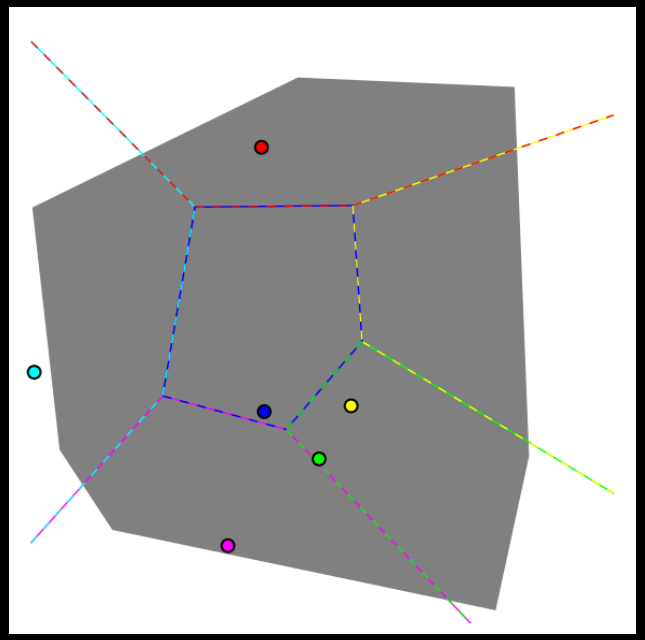

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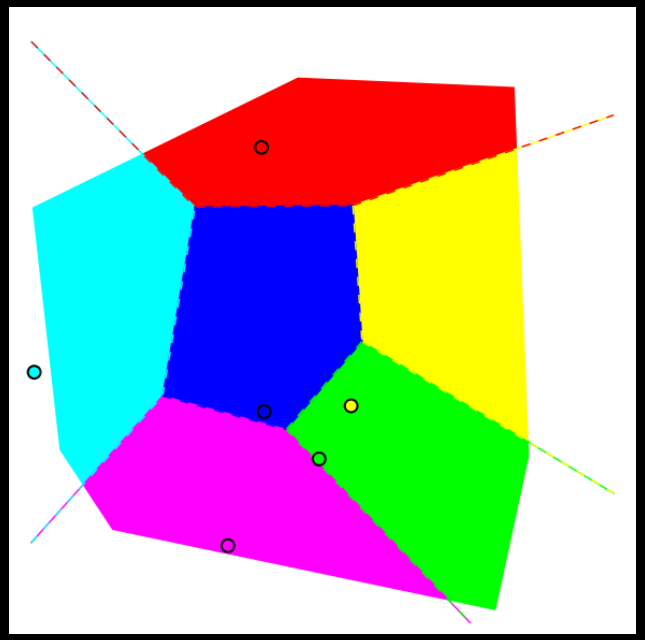

Cutting polygons and the twisted Euler class of $\zeta_{\mathbb{R}^d,k}^{\oplus d-1}$

Voronoi diagrams $\mathcal{V}(x_1,\dots,x_k) = (C_1,\dots,C_k)$

$$\begin{aligned} C_i &:= \{x \in \mathbb{R}^d : \|x - x_i\| \le \|x - x_j\| \text{ for } 1 \le j \le k\} \\ &= \{x \in \mathbb{R}^d : \|x - x_i\|^2 - \|x\|^2 \le \|x - x_j\|^2 - \|x\|^2 \text{ for } 1 \le j \le k\}. \end{aligned}$$

Generalized Voronoi diagrams
$\mathcal{P}(x_1,\dots,x_k; w_1,\dots,w_k) = (P_1,\dots,P_k)$ **where** $w_1,\dots,w_n \in \mathbb{R}$ **and**

$$P_i := \{x \in \mathbb{R}^d : \|x - x_i\|^2 - w_i \le \|x - x_j\|^2 - w_j \text{ for } 1 \le j \le k\}.$$

Theorem (Kantorovich 1939, etc.)

Let μ_d be an absolutely continuous probability measure on $\mathbb{R}^d$ and $k \ge 2$. Then for any $x_1,\dots,x_k \in \mathbb{R}^d$ with $x_i \ne x_j$ for all $1 \le i < j \le h$ there are weights $w_1,\dots,w_h \in \mathbb{R}$, $w_1 + \dots + w_k = 0$, such that the generalized Voronoi diagram $\mathcal{P}(x_1,\dots,x_k; w_1,\dots,w_k)$ equiparts the measure μ_d. Moreover, the weights $w_1,\dots,w_k$

- ***are unique,***
- ***depend continuously on $x_1,\dots,x_k$.***

Cutting polygons and the twisted Euler class of $\zeta_{\mathbb{R}^d,k}^{\oplus d-1}$

K **a convex body in the plane** $\mathbb{R}^2$

$a = p(K \cap P_1) + \cdots + p(K \cap P_k)$

$$\begin{tikzcd}
(x_1, \ldots, x_k) \arrow[r] & (P_1, \ldots, P_k) \arrow[r] & \left(p(K \cap P_1) - \frac{a}{k}, \ldots, p(K \cap P_k) - \frac{a}{k}\right) \\
F(\mathbb{R}^2, k) \arrow[rrr, "\mathfrak{S}_k"'] & & & W_k
\end{tikzcd}$$

Theorem

(1) ***If there is no*** $\mathfrak{S}_k$***-map*** $F(\mathbb{R}^2, k) \longrightarrow_{\mathfrak{S}_k} S(W_k)$***, then the*** k ***Nandakumar & Ramana Rao conjecture has a solution.***

(2) ***If*** $\zeta_{\mathbb{R}^d,k}$ ***does not have a nowhere nonzero cross-section, then there is no*** $\mathfrak{S}_k$***-map*** $F(\mathbb{R}^2, k) \longrightarrow_{\mathfrak{S}_k} S(W_k)$***.***

(3) ***If the twisted Euler class of*** $\zeta_{\mathbb{R}^d,k}$ ***does not vanish, then*** $\zeta_{\mathbb{R}^d,k}$ ***does not have a nowhere nonzero cross-section.***

Cutting polygons and the twisted Euler class of $\zeta_{\mathbb{R}^d,k}^{\oplus d-1}$

Theorem (B., Ziegler, 2014)

$e(\zeta_{\mathbb{R}^d,k}^{\oplus d-1}, \mathcal{Z}^{\otimes d-1}) \neq 0$ ***if and only if*** k ***is a prime power.***

k-regular maps and the dual Stiefel–Whitney class $\bar{w}_{(d-1)(k-\alpha_2(k))}(\xi_{\mathbb{R}^d,k})$

- $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}^N$ **a k-regular ($f\colon \mathbb{C}^d \longrightarrow \mathbb{C}^N$ complex k-regular)**
- $(x_1,\ldots,x_k) \longmapsto (f(x_1),\ldots,f(x_k))$

  $F(\mathbb{R}^d,k) \longrightarrow_{\mathfrak{S}_k} V_k(\mathbb{R}^N)$ $\qquad$ $F(\mathbb{C}^d,k) \longrightarrow_{\mathfrak{S}_k} V_k^{\mathbb{C}}(\mathbb{C}^N)$

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- $(x_1,\ldots,x_k) \longmapsto (f(x_1),\ldots,f(x_k))$

  $F(\mathbb{R}^d,k) \longrightarrow_{\mathfrak{S}_k} V_k(\mathbb{R}^N)$ $\qquad$ $F(\mathbb{C}^d,k) \longrightarrow_{\mathfrak{S}_k} V_k^{\mathbb{C}}(\mathbb{C}^N)$
- **Lemma [Cohen & Handel, 1978]**

  $F(\mathbb{R}^d,k) \longrightarrow_{\mathfrak{S}_k} V_k(\mathbb{R}^N)$ **exists** $\Longleftrightarrow$ $\xi_{\mathbb{R}^d,k}$ **has $(N-k)$-inv.**
- **Lemma**

  $\bar{w}_{(d-1)(k-\alpha_2(k))}(\xi_{\mathbb{R}^d,k}) \neq 0 \Longrightarrow$

  no k-regular map $\mathbb{R}^d \longrightarrow \mathbb{R}^{d(k-\alpha_2(k))+\alpha_2(k)-1}$

Theorem (B., Lück, Ziegler 2013)

If $d \geq 2$***, then*** $\bar{w}_{(d-1)(k-\alpha_2(k))}(\xi_{\mathbb{R}^d,k}) \neq 0$***.***

- $d = 2$***: Cohen & Handel 1978,***
- $d = 2^m$***: Chisholm 1979***

Billiards and $e(\zeta_{\mathbb{R}^d,k}^{\oplus d-1}, \mathcal{Z}^{\otimes d-1})$ for k an odd prime

Problem

Let $d, k \geq 2$ and $T^d \subset \mathbb{R}^d$ be a smooth strictly convex body. Estimate the number of periodic billiard trajectories $N(T^d, k)$ of the period k on the body T^d modulo the action of dihedral group D_{2k}.

- **k is an odd prime**
- $G(S^{d-1}, k) = \{(x_1, \ldots, x_k) \in (S^{d-1})^k : x_i \neq x_{i+1}\}$
- $L\colon G(S^{d-1}, k) \longrightarrow \mathbb{R}$, $(x_1, \ldots, x_k) \longmapsto -\sum |x_i - x_{i+1}|$
- **Lemma**

  $(x_1, \ldots, x_k)$ **periodic traj.** $\Longleftrightarrow$ $(x_1, \ldots, x_k)$ **critical pt. of** L
- $N(T^d, k) \geq \operatorname{cat}\big(G(S^{d-1}, k)/D_{2k}\big)$
- $N(T^d, k) \geq \operatorname{cat}\big(G(S^{d-1}, k)/D_{2k}\big) \geq \operatorname{cat}\big(G(S^{d-1}, k)/\mathbb{Z}/k\big)$
  $\geq max\{\operatorname{wgt}(a) : a \in H^*(G(S^{d-1}, k)/\mathbb{Z}/k; \mathbb{F}_k)$

Billiards and $e(\zeta_{\mathbb{R}^d,k}^{\oplus d-1}, \mathcal{Z}^{\otimes d-1})$ for k an odd prime

$$F(R^{d-1},k)/Z/k \longrightarrow G(S^{d-1},k)/Z/k \longrightarrow \mathrm{B}\,\mathbb{Z}/k$$

$$H^*(F(R^{d-1},k)/Z/k) \longleftarrow H^*(G(S^{d-1},k)/Z/k) \longleftarrow H^*(\mathrm{B}\,\mathbb{Z}/k)$$

$$0 \neq a^* \longleftarrow a$$

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$$0 \neq a^* \longleftarrow a$$

$a^* =$ **(reduction of coefficients)**$\circ(\mathrm{res}_{\mathbb{Z}/k}^{\mathfrak{S}_k})\,(e(\zeta_{\mathbb{R}^d,k}^{\oplus d-1}, \mathcal{Z}^{\otimes d-1}))$

Billiards and $e(\zeta_{\mathbb{R}^d,k}^{\oplus d-1}, \mathcal{Z}^{\otimes d-1})$ for k an odd prime

Base on the work of Farber and Farber & Tabachnikov:

Theorem (Karasev, 2009)

Let $d \geq 3$ ***and*** $k > 2$ ***be a prime. Then***

$$N(T^d, k) \geq (d-2)(k-1) + 2$$

How do we actually compute these classes?

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"Proofs should be communicated only by consenting adults in private."
— Victor Klee (U. Washington)

A glimpse into the proof of $e(\zeta_{\mathbb{R}^d,k}^{\oplus d-1}, \mathcal{Z}^{\otimes d-1}) \neq 0$

$$\begin{tikzcd}
F(\mathbb{R}^d, k) \arrow[r, "\mathfrak{S}_k"'] & S(W_k^{\oplus d-1}) \\
\mathcal{C}(k, d) \arrow[u, "\mathfrak{S}_k"] \arrow[ur, "\mathfrak{S}_k"'] &
\end{tikzcd}$$

A glimpse into the proof of $e(\zeta_{\mathbb{R}^d,k}^{\oplus d-1}, \mathcal{Z}^{\otimes d-1}) \neq 0$

$$F(\mathbb{R}^d, k) \xrightarrow[\mathfrak{S}_k]{} S(W_k^{\oplus d-1})$$

if and only if

$$\gcd\left\{\binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{k-1}\right\} = 1$$

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Theorem (Ram 1909)

For all $n \in \mathbb{N}$ we have

$$\gcd\left\{\binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{k-1}\right\} = \begin{cases} p & \textit{if } k \textit{ is a prime power } p^\ell, \\ 1 & \textit{otherwise.} \end{cases}$$

A glimpse into the proof of $\bar{w}_{(d-1)(k-\alpha_2(k))}(\xi_{\mathbb{R}^d,k}) \neq 0$

$$\overline{w}_{(d-1)(k-1)}$$

$$= \sum_{\substack{j_1,\dots,j_{k-1}\geq 0 \\ j_1+2j_2+\cdots+(k-1)j_{k-1}=(d-1)(k-1)}} \binom{j_1+\cdots+j_{k-1}}{j_1,\ j_2,\ \dots,\ j_{k-1}} w_1^{j_1}\cdots w_{k-1}^{j_{k-1}}$$

$$= w_{k-1}^{d-1} + \sum_{\substack{j_1,\dots,j_{k-2}\geq 0;\ d-2\geq j_{k-1}\geq 0 \\ j_1+2j_2+\cdots+(k-1)j_{k-1}=(d-1)(k-1)}} \binom{j_1+\cdots+j_{k-1}}{j_1,\ j_2,\ \dots,\ j_{k-1}} w_1^{j_1}\cdots w_{k-1}^{j_{k-1}},$$

$$\binom{j_1+\cdots+j_{k-1}}{j_1,\ i_2,\ \dots,\ j_{k-1}} = \frac{(j_1+\cdots+j_{k-1})!}{(j_1)!\cdots(j_{k-1})!}$$

A glimpse into the proof of $\bar{w}_{(d-1)(k-\alpha_2(k))}\big(\xi_{\mathbb{R}^d,k}\big) \neq 0$

$$\overline{w}_{(d-1)(k-1)}$$

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- **Cohen & Handel, 1978,** $d=2$: $\overline{w}_{(d-1)(k-1)} = \overline{w}_{k-1} = w_{k-1} \neq 0$,
- **Chisholm, 1979,** $d=2^m$: $w_{(d-1)(k-1)} \neq 0$ **and** $\frac{\big(j_1+\cdots+j_{k-1}\big)!}{(j_1)!\cdots\big(j_{k-1}\big)!} = 0$, **when all** j_* **are even**
- **B., Lück, Ziegler, 2013,** $d\geq 1$: $w_{(d-1)(k-1)} \neq 0$ **and** $w_1^{j_1}\cdots w_{k-1}^{j_{k-1}} = 0$.

A glimpse into the proof of $\bar{c}_{(d-1)(k-\alpha_p(k))}(\xi_{\mathbb{C}^d,k}) \neq 0$

p **an odd prime**

$d = p^t$ **for** $t \geq 1$

$\bar{c}_{(d-1)(k-\alpha_p(k))}(\xi_{\mathbb{C}^d,k}) \in H^*(F(\mathbb{C}^d, k)/\mathfrak{S}_k; \mathbb{F}_p)$

A glimpse into the proof of $\bar{c}_{(d-1)(k-\alpha_p(k))}(\xi_{\mathbb{C}^d,k}) \neq 0$

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Theorem (B., F. Cohen, Lück, Ziegler, 2015)

Let $d \geq 2$ ***and*** $k \geq 2$ ***be integers, and let*** p ***be an odd prime. Then***

$$\mathrm{height}\,(H^*(F(\mathbb{R}^d, k)/\mathfrak{S}_k; \mathbb{F}_p)) \leq \min\{p^t : 2p^t \geq d\}.$$

the image of $F(M, k)$ under the map defined by a generic set $\{f_1, \dots, f_N\}$ does not intersect this set.

Remark. The estimate of Theorem 2 is not realistic even in the case $k = 2$: the strong Whitney imbedding theorem implies that $I(M^n, 2) \leq 2n + 1$.

Conjecture 2. *If n is a power of 2, then the nth power of any element of positive dimension in $H^*(B(\mathbb{R}^n, k), Z_2)$ equals zero. In particular, $[w(T(\mathbb{R}^n, k))]^n = 1$.*

Conjecture 1 follows (in the same way as Theorem 1) from the latter conjecture and from the fact that if k is a power of 2, then the class $[w_{k-1}(T(\mathbb{R}^n, k))]^{n-1}$ is nontrivial.

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Vassiliev conjecture

Theorem (B., F. Cohen, Lück, Ziegler, 2015)

Let $d \geq 2$ ***and*** $k \geq 2$ ***be integers. Then***

$$\mathrm{height}\left(H^*(F(\mathbb{R}^d, k)/\mathfrak{S}_k; \mathbb{F}_2)\right) \leq \min\{2^t : 2^t \geq d\}.$$

THANK YOU FOR YOUR ATTENTION

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